Geometric Methods of Complex Analysis

October 21-25, 2024, Wuppertal

Segre and Chern forms associated with singular metrics

Tue 10:30-11:20

Mats Andersson

University of Gothenburg, Sweden

Let $E \to X$ be a holomorphic vector bundle. In previous works by Lärkäng, Raufi, Ruppenthal, Sera and Wulcan, is proved that, for a quite wide class of Hermitian metrics on E with analytic singularities, one can associate reasonable Segre forms and Chern forms. They coincide with the usual forms where the metric is smooth and represent the expected Bott-Chern cohomology classes. We pursue this work further and study the residue part in more detail. In particular we prove that all multiplicities (Lelong numbers) of the forms are integers if the singularities are integral analytic. This is a joint work in progress with Richard Lärkäng.

The symplectic density property for Calogero–Moser spaces

Mon 14:30-15:20

Rafael Andrist

University of Ljubljana, Slovenia

A Calogero–Moser space describes the completed phase space of a system of finitely many indistinguishable particles with a certain Hamiltonian in classical physics. Since the past two decades, these spaces are also an object of ongoing study in pure mathematics. In particular, the Calogero–Moser space of n particles is known to be a smooth complex-affine variety equipped with a holomorphic symplectic form, and to be diffeomorphic to the Hilbert scheme of n points in the affine plane.

After extending the Andersen–Lempert theory to the holomorphic symplectic setting, we establish the symplectic density property for the Calogero–Moser spaces and describe their group of holomorphic symplectic automorphisms.

Joint work with Gaofeng Huang.

Picard-type extension theorems for unbounded target domains

Tue 11:30-12:20

Gautam Bharali
Indian Institute of Science, India

Visibility with respect to the Kobayashi distance is a notion of negative curvature that, in recent years, has had many applications in the realm where one needs to control the behaviour of certain classes of holomorphic maps into domains with the visibility property. The word ”visibility” here refers to a property resembling visibility in the sense of Eberlein–O'Neill for Riemannian manifolds. Picard-type theorems are a class of extension theorems for holomorphic maps into a relatively compact complex submanifold embedded in a larger complex manifold in a specific manner, of which the Big Picard Theorem is a special case. We will discuss how the notion of visibility with respect to the Kobayashi distance helps to generalise the latter theorems to the non-relatively-compact setting. This represents joint work with Annapurna Banik.

Complex Dynamics and Complex Analysis

Fri 11:30-12:20

Eric Bedford
Stony Brook University, USA

We will describe some aspects of complex dynamics in complex dimension 2, and we will show the complex analysis/geometry that arises from this.

Invariant subspaces for finite index shifts and the invariant subspace problem in Hilbert spaces

Wed 11:30-12:20

Filippo Bracci
University of Rome Tor Vergata, Italy

The invariant subspace problem, a long-standing open question in operator theory originating from Von Neumann, asks whether every continuous linear operator T on a Hilbert space H (of dimension greater than 1) admits a closed, proper subspace V of positive dimension such that $T(V)$ is contained in V. Despite extensive study and various attempts to solve it, the problem remains unresolved, except in specific cases, such as when H has finite dimension or is non-separable, or when T has defect 1 or less, or is polynomially compact, or has certain special properties. In this talk, I propose a new approach to this problem by leveraging the universality property of backward shifts. Specifically, I will present a characterization of closed subspaces in finite direct sums of Hardy spaces H^2 on the unit disc that are invariant under the shift operator, thereby extending the classical Beurling theorem. This characterization allows us to demonstrate that finite-index shifts do not have maximal non-trivial closed invariant subspaces. Through the universality of backward shifts, this result further implies that the invariant subspace problem has a positive answer for any operator with finite defect. The talk is based on a joint work with Eva Gallardo-Gutierrez.

Runge and Mergelyan theorems in families of open Riemann surfaces

Wed 09:00-09:50

Franc Forstnerič

University of Ljubljana, Slovenia

We prove the Runge and Mergelyan approximation theorems for any continuous or smooth family of complex structures on an open orientable smooth surface, with continuous or smooth dependence of the data and the approximating holomorphic functions on the complex structure. We then develop the Oka theory for maps from families of open Riemann surfaces to any Oka manifold. Proofs depend on the existence of smooth families of quasiconformal deformations of the identity map on relatively compact smoothly bounded domains in open Riemann surfaces.

On the p-Poincaré inequality

Mon 16:00-16:50

Anne-Katrin Gallagher

Gallagher Tool & Instrument, USA

I will talk about the equivalence of the validity of the p-Poincaré inequality on an open set Ω in $\mathbb{R}^n$, i.e.,

$$\|f\|_{p,\Omega} \leq C\|\nabla f\|_{p,\Omega} \quad \text{for all } f \in \mathcal{C}_c^\infty(\Omega)$$

for some $C > 0$, and the finiteness of the p-capacitary inradius, $\rho_p(\Omega)$, of Ω defined by

$$\rho_p(\Omega) = \sup\{r > 0 : \forall \epsilon > 0 \ \exists x \in \mathbb{R}^n \text{ such that } C_p(\overline{\mathbb{B}_r(x)} \cap \Omega^c) < \epsilon\}.$$

Here, $C_p(E)$ denotes the Sobolev p-capacity of $E \subset \mathbb{R}^n$.

Positivity of boundary intersections of J-complex curves

Mon 10:30-11:20

Sergey Ivashkovich

University of Lille, France

Let an almost complex structure J of class $\mathcal{C}^1$ in a neighborhood of the origin in $\mathbb{R}^{2n}$ be given, $n \geq 2$. Let furthermore $0 \in W$ be a germ of a J-totally real submanifold of real dimension n and of class $\mathcal{C}^2$. After an appropriate $\mathcal{C}^2$-smooth coordinate change we can assume wlg that for an appropriate neighborhood B of zero we have that $(B, W) = (\mathbb{R}^{2n}, \mathbb{R}^n)$ and $J|_{\mathbb{R}^n} = J_{st}$. Here J_{st} stands for the standard complex structure of $\mathbb{C}^n$. Such coordinate change Ψ we shall call a redressing map. Denote by $\Delta^+ := \{\zeta \in \Delta : \Im\zeta \geq 0\}$ the upper half-disk and by $\beta_0 := (-1, 1)$ its edge. Let a J-holomorphic map $u : (\Delta^+, \beta_0, 0) \to (\mathbb{R}^{2n}, \mathbb{R}^n, 0)$ be given, assume it is continuous up to β_0. We shall call such u a J-complex half-disk attached to W. It is not difficult to prove that there exist a $\mu \in \mathbb{N}$ such that

$$u(\zeta) = v_0\zeta^\mu + O(|\zeta|^{\mu+\alpha}) \quad \text{with} \quad v_0 \neq 0. \tag{1}$$

We shall call v_0 the tangent vector to u at zero and (1) the normal form of a J-complex half-disk attached to a totally real submanifold. Number μ we shall call the order of vanishing of u at zero. This μ doesn't depend on the redressing map Ψ. Making a reflection with respect to $W = \mathbb{R}^n$, i.e., setting

$$\tilde{u}(\zeta) = \begin{cases} u(\zeta) \text{ if } \Im\zeta \geq 0 \\ \overline{u(\bar{\zeta})} \text{ if } \Im\zeta < 0, \end{cases} \tag{2}$$

we can extend u to Δ as a $\mathcal{C}^\alpha$-regular map. Now let $u_1, u_2 : (\Delta^+, \beta_0, 0) \to (\mathbb{R}^4, \mathbb{R}^2, 0)$ be two J-complex disks attached to W which intersect at zero. We define the boundary intersection index of u_1 and u_2 at zero as the intesection number at zero of their extensions by reflection, and denote this index as $\mathsf{ind}_0^b(u_1, u_2)$. Our goal is present the following

Theorem 1. *Let J be a $\mathcal{C}^1$-regular almost-complex structure on $\mathbb{R}^{2n}$ such that $J|_{\mathbb{R}^n} = J_st$ and let $u_i : (\Delta^+, \beta_0, 0) \to (\mathbb{R}^{2n}, \mathbb{R}^n, 0)$ be two J-complex half-disks such that one is not a reparameterization of another. Then their boundary intersection index is correctly defined and satisfies*

$$\mathsf{ind}_0^b(u_1, u_2) \geq \mu_1 \cdot \mu_2, \tag{3}$$

where μ_i is the order of vanishing of u_i at zero, $i = 1, 2$.

Invariant metrics and rescaling sequences

Tue 16:00-16:50

Jaecheon Joo

King Fahd University of Petroleum and Minerals, Saudi Arabia

The idea of rescaling is to make a (divergent) sequence of mappings converge by composing with linear affine maps. The method is used in wide range of mathematics and in Complex Analysis, S. Frankel (1989, Acta Math.) also used his version of rescaling sequence in order to characterize convex domains which are universal covering space of compact complex manifolds. In this talk, I would like to introduce a geometric way to interpret Frankel's sequence and a condition for the convergence. I will also introduce some open questions related with the geometric condition and the symmetry of the limit.

Holomorphic mappings and Topological invariants

Mon 11:30-12:20

Kang-Tae Kim
POSTECH, South Korea

I wish to discuss invariants for Riemann surfaces covered by the disc and for hyperbolic manifolds in general involving measure of the image over the homotopy and homology classes of closed curves and maps of the k-sphere into the manifold. The invariants are monotonic under holomorphic mappings and strictly monotonic under certain circumstances. Applications to holomorphic maps of annular regions in $\mathbb{C}$ and tubular neighborhoods of compact totally-real submanifolds in general in $\mathbb{C}^n, n \geq 2$, will also be discussed. I will also discuss the contractibility of a hyperbolic domain with contracting holomorphic mapping.

This talk is based on [R. E. Greene, K.-T. Kim and N. V. Shcherbina: Topological invariants and holomorphic mappings. *C. R. Math.* (France). 2022, Vol. 360, p. 829–844. `https://doi.org/10.5802/crmath.336`]

Hölder continuous solutions of complex Hessian equations

Wed 10:30-11:20

Sławomir Kołodziej
Jagiellonian University, Poland

This is joint work with Cuong Ngoc Nguyen. First we prove that the Dirichlet problem for the complex Hessian equation has the Hölder continuous solution provided it has a subsolution with this property.

In the second part we develop the theory of weak solutions for this kind of equations on compact Hermitian manifolds.

Applications of Complex Analysis to the theory of Second Order ODEs

Tue 14:30-15:20

Ilya Kossovskiy
Masaryk University, Czech Republic

In our recent research, we have discovered interactions between Complex Analysis and the theory of completely integrable systems of PDEs/ODEs. This has recently allowed for solving several difficult problems in Complex Analysis (CR geometry) by using methods of Complex Dynamics. In this lecture, I will outline applications of the method in the other direction. In particular, I will explain our work with Zaitsev on the complete normal form for second order ODEs. Then I will outline a more recent result of us on the low regularity version of Cartans-Tresse's flatanning theorem for cubic second order ODEs.

Invariant curve in strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$

Thu
10:30-
11:20

Bernhard Lamel
University of Vienna, Austria

We discuss the relationship between the following curves: Chern-Moser chains, boundaries of extremal discs, and traces of Segre varieties (the later only defined for real-analytic boundaries of course). It turns out that if any of these families coincide, then the hypersurface is locally spherical. For traces of Segre varieties and chains this result is due to Faran, the equivalence of the other families is joint work with Giuseppe Della Sala and Florian Bertrand (both American University of Beirut).

On Stein neighborhoods of certain Stein domains

Mon
09:00-
09:50

Takeo Ohsawa
Nagoya University, Japan

Let X be a complex manifold which is holomorphically embedded in a complex manifold U as a closed submanifold. A well-known theorem of Siu [S] says that X admits a Stein neighborhood system in U if X is Stein.

In view of Siu's theorem, it may be worthwhile to study the existence of smoothly bounded Stein neighborhoods of X when U is a bounded domain in a complex manifold M and $X = U \cap Y$ for some complex submanifold Y of M. Such a question is closely related to the study of Dirichlet problem and $\bar{\partial}$-Neumann problem. Siu's method may be regarded as a natural extension of the observation that one can find a smoothly bounded strongly pseudoconvex neighborhood system of X in M if X is a strongly pseudoconvex domain in Y.

The purpose of this talk is to outline a proof of the following.

Theorem 1. *There exists a two-dimensional compact complex manifold Y holomorphically embedded in $\mathbb{CP}^5$ and a Stein domain $X \subset Y$ with real-analytic boundary such that X admits a Stein neighborhood system with a "log-Hölder continuous" boundary in $\mathbb{CP}^5$ but does not admit a Stein neighborhood system with Lipschitz continuous boundary.*

In this example, X is the domain defined by

$$\{(\zeta, [z]); \mathrm{Re}(\zeta e^{2\pi i \mathrm{Re} z}) < 0\}$$

in the Veronese embedded product of $\mathbb{CP}^1 = \{\zeta \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\}$ and an elliptic curve $A = \{[z] = z + \mathbb{Z} + i\mathbb{Z}; z \in \mathbb{C}\}$. For the proof of the first part, Siu's proof is refined by the L^2 method. The latter part is an application of Harrington's result [H] asserting that Lipschitz pseudoconvex domains in $\mathbb{CP}^n$ admit bounded plurisubharmonic exhaustion functions, which is a refinement of Takeuchi's theorem [T] as well as [Oh-S]. A similar construction was given in [D-Oh] which does not assert the Hölder continuity.

References

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The Independence Polynomial on arbitrary recursive graphs

Fri 09:00-09:50

Han Peters

University of Amsterdam, Netherlands

In the last decade we saw several remarkable papers discussing properties of graph polynomials on recursive graphs. These graph polynomials include the partition functions of Ising, Potts, and Hard-Core model, the chromatic polynomial, the independence polynomial and several others. The recursive graphs include hierarchical lattices, Cayley trees, Sierpinsky triangles and many other examples of Schreier graphs.

The main advantage of working with these recursive sequences of graphs is that the recursions corresponds to some iterative procedure on the level of the partition functions, often given in terms of rational maps in one or several (complex) variables. By studying the dynamical behavior of these systems, properties of the graph polynomials such as phase transitions and computational hardness were successfully described.

In current work with Mikhail Hlushchanka (University of Amsterdam) we attempt to set up a general framework for sequences of recursive graphs in the setting of the independence polynomial. We consider an arbitrary starting graph with k marked vertices, and at each step construct a new graph by taking n copies of the previous graph and joining these copies along marked vertices according to a fixed algorithm. The dynamical systems that arise are given by homogeneous polynomials of degree n in 2^k variables. Somewhat surprisingly it turns out that these systems can be successfully described in this general setting.

A new approach to the study of m-convex functions

Thu 09:00-09:50

Azimbay Sadullaev
National University of Uzbekistan, Uzbekistan

The report is devoted to m-convex $(m-cv)$ functions in Euclidean space $\mathbb{R}^n$, $1 \leq m \leq n$. When $m=n$ the class n-cv coincides with the class of subharmonic functions $sh = \{\lambda_1+\lambda_2+\ldots+\lambda_n \geq 0\}$, and when $m=1$ it coincides with the class of convex functions $cv = \{\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_n \geq 0\}$:

$$cv = 1-cv \subset 2-cv \subset \ldots \subset n-cv = sh.$$

The theory of subharmonic functions is developed and important part of the Theory of Functions and Mathematical Physics. The theory of Convex Functions is well studied and reflected in the works of A. Aleksandrov, I. Bakelman, A. Pozdnyak, V. Pogorelov, etc. For $m>1$ this class was studied in a series of works by N. Ivochkina, N. Trudinger, H. Wong, S.Y. Lee et al.

We propose a new approach to the study of $m-cv$ functions, establishing their connection with m-subharmonic (sh_m) functions in complex space $\mathbb{C}^n$. The theory of sh_m-functions is based on differential forms and currents $(dd^cu)^k \wedge \beta^{n-k} \geq 0$, $k=1,2,\ldots,n-m+1$, where $\beta = dd^c||z||^2$ – the standard volume form in $\mathbb{C}^n$ and is well developed Z. Blocki, [Bl], S. Dinev and S. Kolodziej [DK], S. Li [Li], H.Ch. Lu [Lu1], [Lu2], etc.). A fairly complete review of this theory is available in the article by A. Sadullaev and B. Abdullaev in the Proceedings of MIRAN.

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On the polynomially convex embedding dimension

Thu 14:30-15:20

Håkan Samuelsson

University of Gothenburg, Sweden

If M is an arbitrary smooth compact real n-dimensional manifold, what is the smallest integer N such that M can be smoothly embedded into $\mathbb{C}^N$ as a polynomially convex set? This natural question was asked by Izzo and Stout in 2018. A related question is how many smooth functions on M it takes to generate the uniform algebra of all continuous functions on M. An upper bound follows by classical works by Forstnerič and Rosay. This bound was recently improved by Gupta and Shafikov. I will present a joint work in progress with Arosio and Wold to answer the above questions when $n \leq 11$.

Realizing semisimple Lie groups as automorphism groups of bounded domains

Thu 11:30-12:20

Alexander Tumanov

University of Illinois, USA

For a bounded domain D in $\mathbb{C}^n$, the group $\mathrm{Aut}(D)$ of all biholomorphic automorphisms of D is a (finite dimensional) Lie group (H. Cartan, 1935). Is the converse true? Bedford and Dadok (1987) and Saerens and Zame (1987) proved that every compact Lie group G can be realized as $\mathrm{Aut}(D)$ for a bounded strongly pseudoconvex domain D in $\mathbb{C}^n$. When the group G is not necessarily compact, we proved (1990) that every connected linear Lie group G can be realized as $\mathrm{Aut}(D)$, where D is a (possibly unbounded) strongly pseudoconvex domain in $\mathbb{C}^n$ of bounded type. When the group G is not necessarily linear, Winkelman (2004) and Kan (2007) proved that G can be realized as $\mathrm{Aut}(D)$, where D is a hyperbolic Stein manifold. The question whether D can be chosen a bounded domain in $\mathbb{C}^n$ has remained largely open. Recently, we proved the result for some examples of non-linear Lie groups, including the universal cover of $\mathrm{SL}(2, R)$. We now extend the result to all connected semisimple Lie groups. This work is joint with George Shabat.

Multiplier ideal sheaves, curvature positivity, and L^2 estimates for $\bar{\partial}$-equation

Tue 09:00-09:50

Xiangyu Zhou

Chinese Academy of Sciences, China

In this talk, we'll first recall some basic and recent results on multiplier ideal sheaves associated to plurisubharmonic functions, including Guan-Zhou's solution of Demailly's strong openness conjecture, and then introduce some new results on multiplier submodule sheaves which are the vector bundle version of multiplier ideal sheaves. We'll also present our result on characterizing Nakano positivity via solving $\bar{\partial}$-equations with optimal L^2-estimates (established by Deng-Ning-Wang-Zhou), which is a converse proposition of Hörmander-Demailly's L^2 existence theorems, which establishes a connection between differential geometry and partial differential equations via several complex variables. As an application of the criterion, we give an affirmative answer to Lempert's problem (solved by Liu-Yang-Zhou), which asks whether the limit metric of an increasing sequence of hermitian metrics with Nakano semi-positive curvature on holomorphic vector bundles is still Nakano semi-positive.

Burns-Krantz rigidity in non-smooth domains

Fri 10:30-11:20

Włodzimierz Zwonek

Jagiellonian University, Poland

We present a method that enables to show a boundary Schwarz-Lemma for non-smooth domains. The method allows to show the Burns-Krantz rigidity type theorem in domains such as the polydisc or the symmetrized bidisc and is based on the study of regular complex geodesics and their left inverses and applies to domains where the Lempert theorem holds.
