Multiplier ideal sheaves, curvature positivity and L^2 estimates for $\bar{\partial}$ equation

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Conference on Geometric Methods of Complex Analysis
Wuppertal University

Oct. 22, 2024

psh functions, introduced by Oka and Lelong, plays a fundamental role in several complex variables/complex geometry

- singularity of a psh φ : $\varphi(z) = -\infty$
  e.g., for $\varphi = c\ log(|f_1|^2 + \cdots + |f_k|^2)$ is psh, where $c > 0$
- relation between a complex analytic subset and a complete pluripolar set :
  $f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) = \varphi^{-1}(-\infty)$
- special case : Kähler form (metric), local potential (smooth strictly psh)
- psh with (neat) analytic singularities :
  $\varphi = c\ log(|f_1|^2 + \cdots + |f_k|^2)$ up to a bounded function (smooth function)

- singular hermitian metric on a holomorphic line bundle : locally $e^{-\varphi}, \varphi \in L^1_{loc}$
  curvature $\Theta = i\partial\bar{\partial}\varphi$ in the sense of currents
  • pseudoeffective line bundle : $\Theta \geq 0$, i.e., φ is psh.
  • big line bundle : the curvature current $\Theta \geq \epsilon\omega$ for some $\epsilon > 0$
  • positive line bundle : φ is smooth strictly psh
- psh and positive closed (1,1)-current :
  • φ is psh, then $i\partial\bar{\partial}\varphi$ is a positive closed (1,1)-current
  • a positive closed (1,1)-current has locally a psh potential

▶ complex Monge-Amperé : pluripotential theory

▶ singular Kähler-Einstein metric : ω-plurisubharmonic functions

• ω-psh functions on Hermitian manifold (M, ω)

• the completion of the definition domains of Mabuchi functional, Ding functional, K-energy,...

equivalence of singularities

- φ is more singular than ψ, denoted by $\varphi \preccurlyeq \psi$: if $\varphi \leq \psi + O(1)$
- φ and ψ have equivalent singularities : if $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$

classical invariants :

- Lelong number : $v(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{ln|z-x|}$
- complex singularity exponent (log canonical threshold) $c_x(\varphi) = sup\{c \geq 0 : \exp^{-2c\varphi}$ is L^1 w.r.t. the Lebesgue measure on $\mathbb{C}^n$ on a neighborhood of $x\}$

S.T. Yau : the notion of multiplier ideal sheaf plays a central role in modern higher-dimensional algebraic geometry

Y-T. Siu : L^2 extension and multiplier ideal sheaves are main ingredients of solving important conjectures in alg. geom.

- **Definition of multiplier ideal sheaf :**

to a plurisubharmonic function φ, is associated an ideal subsheaf $\mathcal{I}(\varphi)$ of $\mathcal{O}$: germs of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 e^{-\varphi}$ is locally integrable at x.

- to pseudoeffective line bundle $(L, h = e^{-\varphi})$, is associated a multiplier ideal sheaf $\mathcal{I}(h)$ consists of germs of holomorphic sections f of L s.t. $|f|_h^2 = |f|^2 e^{-\varphi}$ is locally integrable

- L^p multiplier ideal sheaf : $|f|^p e^{-\varphi}$ is locally integrable.

• origin goes back to Hörmander, Bombieri, Skoda : **Hörmander-Bombieri-Skoda theorem**, an early result on L^2 extension problem :

let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $\varphi \in \mathrm{psh}(\Omega)$, if $e^{-\varphi}$ is integrable on $\mathbb{B}^n(z_0; r) \subset\subset \Omega$, then there exists a holomorphic function $f \in \mathcal{O}(\Omega)$ so that $f(z_0) = 1$ and $\int_\Omega |f(z)|^2 e^{-\varphi}(r^2 + |z|^2)^{-n-\varepsilon} d\lambda < +\infty$.

Basic Properties

- Nadel theorem : $\mathcal{I}(\varphi)$ is coherent
  • non local integral point set of $e^{-\varphi}$
  = the zero set of $\mathcal{I}(\varphi)$
  = *supp* $\mathcal{O}/\mathcal{I}(\varphi)$ is an analytic set
- Theorem : multiplier ideal sheaf is integrally closed, i.e., the integral closure of $\mathcal{I}(\varphi)$ is itself
- Nadel vanishing theorem : Let $(L, e^{-\varphi})$ be a big line bundle on a weakly pseudoconvex Kähler manifold X. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

  for any $q \geq 1$.

the Kähler condition in Nadel vanishing theorem is not necessary for holomorphically convex manifolds.

Theorem (Meng, Zhou, JAG 2019)
Let (X, ω) be a compact Hermitian holomorphically convex manifold, and let L be a big holomorphic line bundle over X equipped with a singular Hermitian metric h. Assume that $i\Theta_{L,h} \geq \varepsilon\omega$ for some continuous positive function ε on X. Then

$$H^q(X, \mathscr{O}(K_X \otimes L) \otimes \mathcal{I}(h)) = 0 \quad for \quad q \geq 1$$

.

• By Nadel vanishing theorem , multiplier ideal sheaves could be used to give a unified solution to

- Levi problem (Oka, Grauert) in several complex variables and
- Kodaira embedding theorem in algebraic geometry

Denote by

$$\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi) = \mathcal{I}((1+\varepsilon_0)\varphi).$$

Demailly's strong openness conjecture :
For any plurisubharmonic function φ on X, one has

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

• J-P. Demailly, Multiplier ideal sheaves and analytic methods in algebraic geometry. School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 6, Trieste, 2001.

The meaning of the conjecture :

✓ $|f|^2e^{-\varphi}$ is locally integrable, then there exists an $\varepsilon_0 > 0$ s.t. $|f|^2e^{-(1+\varepsilon_0)\varphi}$ is also locally integrable

✓ $\{p \in \mathbb{R} : |f|^2e^{-p\varphi}$ is locally integrable$\}$ is open

origin from calculus :

- $\{p \in \mathbb{R} : 1/|x|^{pc} = e^{-p\varphi}$ is locally integrable at the origin$\}$ is open, where $\varphi = c\ log|x|, c > 0$
- $\{p \in \mathbb{R} : f(x)/|x|^{pc} = f(x)e^{-p\varphi}$ is locally integrable at the origin$\}$ is open

✓ also conjectured by Y.T. Siu, Demailly-Kollár, ...

• Y.-T. Siu, Invariance of plurigenera and torsion-freeness of direct image sheaves of pluricanonical bundles. Finite or infinite dimensional complex analysis and applications, 45-83, Adv. Complex Anal. Appl., 2, Kluwer Acad. Publ., Dordrecht, 2004.

• J.-P. Demailly, Analytic Methods in Algebraic Geometry, Higher Education Press, Beijing, 2010

• J-P. Demailly, J. Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Ann. Sci. École Norm. Sup.* (4) 34 (2001), no. 4, 525–556.

✓ Openness conjecture : under the assumption that $e^{-\varphi}$ is locally integrable (i.e., when $\mathcal{I}(\varphi)$ is trivial), it was proved by Berndtsson

(arXiv :1305.5781, published in *Proc. Abel Symposium* 2013)

✓ $dimX = 2$, proved by Favre, Jonsson, Mustaţă

- C. Favre and M. Jonsson, *Invent. Math.* and *JAMS* 2005 ;
- M. Jonsson and M. Mustaţă, *Ann. de L'Inst. Fourier*, 2012.

Theorem. (Guan, Zhou)
Demailly's strong openness conjecture holds.

other forms :

$$\cup_j \mathcal{I}(\varphi_j) = \mathcal{I}(\varphi), \varphi_j \nearrow \varphi$$

or

$$\cup_{\varepsilon>0} \mathcal{I}((\varphi + \varepsilon\psi)) = \mathcal{I}(\varphi)$$

• Q.A. Guan, X.Y. Zhou, Strong openness conjecture for plurisubharmonic functions, arXiv :1311.3781.
• Q.A. Guan, X.Y. Zhou, A proof of Demailly's strong openness conjecture, Ann. of Math.(2) 182 (2015), no. 2, 605–616.
• one of the main ingredients of the proof
✓ a use movably of Ohsawa-Takegoshi's L^2 extension theorem : subvarieties may be not fixed, but movable

Let $f \in \Gamma(D, \mathcal{I}(\varphi))$ and $\varphi_j, \varphi \in \mathrm{PSH}(D)$ for $j = 1, 2, \ldots$

actually, the strong openness conjecture implies the following :

Corollary Assume $\varphi_j \leqslant \varphi$ and $\varphi_j \to \varphi$ increasingly a.e. Then
$|f|^2 e^{-\varphi_j} \to |f|^2 e^{-\varphi}$ in L^1 locally

$\varphi_j \leqslant \varphi_{j+1}$
By the strong openness, there exists j_0 such that $\mathcal{I}(\varphi_{j_0}) = \mathcal{I}(\varphi)$ on $D' \subseteq\subseteq D$, in particular

$$\int_{D'} |f|^2 e^{-\varphi_{j_0}} d\lambda < \infty.$$

Since $|f|^2 e^{-\varphi_j} \leqslant |f|^2 e^{-\varphi_{j_0}}$ for $j \geqslant j_0$, we're done by DCT.

Corollary : General Kawamata-Viehweg-Nadel type vanishing theorem :
Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n, and $nd(L, \varphi)$ be the numerical dimension of (L, φ).

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $q \geq n - nd(L, \varphi) + 1$.

• This was conjectured by Junyan Cao in Compositio Math. 2014. It was originally asked by Demailly

• the result holds by combining Cao's result with strong openness

Corollary : relation between invariants for psh singularities

- $\mathcal{I}(c\varphi) = \mathcal{I}(c\psi)$, for any $c > 0$
- Lelong numbers up to proper modifications are the same : for all proper modifications $\pi : X \to \mathbb{C}^n$ above 0 and all points $p \in \pi^{-1}(0)$, we have $v(\varphi \circ \pi, p) = v(\psi \circ \pi, p)$

• This was conjectured by Boucksom-Favre-Jonsson

• true by combining Boucksom-Favre-Jonsson's result with strong openness

Corollary : For a big line bundle L, the equality $\mathcal{I}(||mL||) = \mathcal{I}(h_{min}^m)$ holds for every integer $m > 0$.

- conjectured by Demailly, Ein, and Lazarsfeld.
- also listed as an open problem in the book :

Lazarsfeld : Positivity in algebraic geometry, Vol 2.

▶ Def. asymptotic multiplier ideal $\mathcal{I}(||L||)$:
maximal member of the family of the ideals $\{\mathcal{I}(1/k \cdot |kL|)\}$ for k large

Theorem (Demailly) : If L is a pseudo-effective line bundle, then there exists a unique (up to equivalence) singular Hermitian metric h_{min} with minimal singularities.

Corollary (Guan-Zhou, Fornaess) : Strong openness also holds for L^p multiplier ideal sheaves for $0 < p < \infty$

- follows from the strong openness for L^2 multiplier ideal sheaves (see Fornaess, Lecture notes at Peking Univ. arXiv)

Siu's Conjecture : strong openness of L^p multiplier ideal sheaves for $1 \leq p < \infty$ holds

Corollary : L^p multiplier ideal sheaves are coherent

- Siu : strong openness implies coherence
- Cao : for $p = 1/m$

✓ strong openness is used essentially in important papers to give new approaches to Yau-Tian-Donaldson conjecture : on Fano manifolds, existence of Kähler-Einstein metric ⇔ K-stability

Berman, Boucksom, Jonsson : A variational approach to the Yau-Tian-Donaldson conjecture. *J. Amer. Math. Soc.* (2021)
Main results essentially rely on the strong openness

Darvas-Zhang discuss a general setting : X is compact Kähler, $-K_X$ is big
a starting point : λ-Ding functional, while the existence of $\lambda > 0$ is guaranteed by the strong openness
the strong openness is repeatedly used in their paper

Theorem (Guan, Li, Zhou) : Stability of the multiplier ideal sheaves holds :
Let $(\varphi_j)_{j\in\mathbb{N}^+}$ be a sequence of negative psh functions on D, which converges to $\varphi \in Psh(D)$ in Lebesgue measure, and $\mathcal{I}(\varphi_j)_o \subset \mathcal{I}(\varphi)_o$. Let $F_j \in \mathcal{O}(D)$, $j \in \mathbb{N}^+$ s.t. $(F_j, o) \in \mathcal{I}(\varphi)_o$, which compactly converges to $F \in \mathcal{O}(D)$. Then, $|F_j|^2 e^{-\varphi_j}$ converges to $|F|^2 e^{-\varphi}$ in the L^1 norm near o.

In particular, there exists $\varepsilon_0 > 0$ such that
$\mathcal{I}(\varphi_j)_o = \mathcal{I}((1+\varepsilon_0)\varphi_j)_o = \mathcal{I}(\varphi)_o$ for any large enough j.

- Guan, Li, Zhou, arXiv :1603.05733. Chinese Ann. of Math. 2022
- based on Guan-Zhou (Invent. Math. 2015)

Corollary (Demailly-Kollár, Hiep) : semi-continuity of complex singularity exponents and weighted lct holds

- Hörmander L^2 method using psh weights :

psh weight regarded as a singular metric on (trivial) pseudoeffective line bundles

Hömander's L^2 existence theorem :

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, φ be a plurisubharmonic function on D, then one can solve $\bar{\partial}u = v$ (where $\bar{\partial}v = 0$) with L^2 estimate $||u||_\varphi := \int_D |u|^2 e^{-\varphi} \leq C||v||_\varphi$, for some constant $C > 0$

• Let (X, ω) be a Hermitian manifold of dimension n, and (E, h) be a Hermitian holo. vector bundle of rank r over X.

• Let $D = D' + \bar{\partial}$ be the Chern connection of (E, h), and $\Theta_{E,h} = [D', \bar{\partial}] = D'\bar{\partial} + \bar{\partial}D'$ be the Chern curvature tensor.

• Denote by $(e_1, \cdots, e_r)$ an orthonormal frame of E over a coordinate patch $\Omega \subset X$ with complex coordinates $(z_1, \cdots, z_n)$, and

$$i\Theta_{E,h} = i \sum_{1 \le j,k \le n, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu, \quad \bar{c}_{jk\lambda\mu} = c_{kj\mu\lambda}.$$

• To $i\Theta_{E,h}$ corresponds a natural Hermitian form $\theta_{E,h}$ on $TX \otimes E$ defined by

$$(1) \qquad \theta_{E,h}(u, u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(x) u_{j\lambda} \bar{u}_{k\mu}, \qquad u \in T_xX \otimes E_x.$$

- E is said to be Nakano positive (resp. Nakano semi-positive) if $\theta_{E,h}$ is positive (resp. semi-positive) definite as a Hermitian form on $TX \otimes E$, i.e. for every $u \in TX \otimes E$, $u \neq 0$, we have

$$\theta(u,u) > 0 \quad (\text{resp. } \geq 0).$$

- E is said to be Griffiths positive (resp. Griffiths semi-positive) if for any $x \in X$, all $\xi \in T_xX$ with $\xi \neq 0$, and $s \in E_x$ with $s \neq 0$, we have

$$\theta(\xi \otimes s, \xi \otimes s) > 0 \quad (\text{resp. } \geq 0).$$

- Nakano negative (semi-negative) and Griffiths negative (semi-negative) are similarly defined by replacing > 0 (resp. ≥ 0) with < 0 (resp. ≤ 0) in the above.

Lemma : Let (X, ω) be a Kähler manifold, (E, h) be a Hermitian holomorphic vector bundle over X. Then (E, h) is Nakano positive (resp. semo-positive) if and only if the Hermitian operator $[i\Theta_{E,h}, \Lambda_\omega]$ is positive definite (resp. semi-positive definite) on $\Lambda^{n,1}T_X^* \otimes E$.

$$\langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda} \bar{u}_{k\mu} \tag{2}$$

where $u = \sum u_{j\lambda} dz \wedge d\bar{z}_j \otimes e_\lambda \in \Lambda^{n,1}T_X^* \otimes E$, and $dz = dz_1 \wedge \cdots \wedge dz_n$.

Demailly's L^2 existence theorem :

Let (X, ω) be a complete Kähler manifold, with a Kähler metric ω which is not necessarily complete. Let (E, h) be a Hermitian vector bundle of rank r over X, and assume that the curvature operator $B := [i\Theta_{E,h}, \Lambda_\omega]$ is semi-positive definite everywhere on $\Lambda^{p,q}T_X^* \otimes E$, for some $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T_X^* \otimes E)$ satisfying $\bar{\partial} g = 0$ and $\int_X \langle B^{-1}g, g\rangle dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T_X^* \otimes E)$ such that $\bar{\partial} f = g$ and

$$\int_X |f|^2 dV_\omega \leq \int_X \langle B^{-1}g, g\rangle dV_\omega.$$

with my former students, we have initiated studying of the converse aspects of L^2 existence and L^2 extension.

- F. Deng, Z. Wang, L. Zhang, and X. Zhou. New characterization of plurisubharmonic functions and positivity of direct image sheaves. *arXiv :1809.10371*. American J. Math. 2024
- Fusheng Deng, Jiafu Ning, Zhiwei Wang, and Xiangyu Zhou. Positivity of holomorphic vector bundles in terms of L^p-conditions of $\bar{\partial}$, 2020. *arXiv :2001.01762v1*. Math. Ann. 2023

• G. Hosono and T. Inayama. A converse of Hörmander's L^2-estimate and new positivity notions for vector bundles. Sci. China Math. 2021

Theorem (Deng-Wang-Zhang-Zhou, Amer. J. Math. 2024)
Let $\varphi : D \to [-\infty, +\infty)$ be an upper semicontinuous function on $D \subset \mathbb{C}^n$ that is not identically $-\infty$. Let $p > 0$ be a fixed constant. If for any $z_0 \in D$ with $\varphi(z_0) > -\infty$ and any $m > 0$, there is $f \in \mathcal{O}(D)$ such that $f(z_0) = 1$ and

$$\int_D |f|^p e^{-m\varphi} \leq C e^{-m\varphi(z_0)},$$

where C is a constant, then φ is plurisubharmonic.

Theorem (Deng-Wang-Zhang-Zhou, AJM)
h maybe singular Finsler. If (E, h) satisfies the optimal or the multiple coarse L^2-extension condition, then (E, h) is Griffiths semi-positive.

in the case of line bundles, $h = e^{-\varphi}$, then φ is psh

Recall a notion introduced by Deng-Ning-Wang-Zhou (Math. Ann. 2023).

- We say (E,h) is L^2 optimal (or of Skoda-Demailly type), if for any Stein coordinate U such that $E|_U$ is trivial, any Kähler form ω_U on U and any $\psi \in \mathrm{Spsh}(U) \cap C^\infty(U)$ such that for any $\bar\partial$-closed $f \in L^2_{(n,1)}(U, \omega_U, E|_U, he^{-\psi})$, there exists $u \in L^2_{(n,0)}(U, \omega_U, E|_U, he^{-\psi})$ such that $\bar\partial u = f$ and

$$\int_U |u|^2_{\omega_U,h} e^{-\psi} dV_{\omega_U} \leq \int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U,h} e^{-\psi} dV_{\omega_U} \tag{3}$$

provided that the right hand is finite, where $B_{\omega_U,\psi} = [\sqrt{-1}\partial\bar\partial\psi \otimes \mathrm{Id}_E, \Lambda_\omega]$.

Theorem

(Hörmander, Demailly, Deng-Ning-Wang-Zhou)
If h is C^2 smooth, then (E, h) is L^2 optimal if and only if h is Nakano semi-positive.

an application : a solution of Lempert's problem

- Zhuo Liu, Hui Yang, and Xiangyu Zhou. On the multiplier submodule sheaves associated to singular Nakano semi-positive metrics. *arXiv :2111.13452*, 2021.

Lempert's problem : whether a C^2 hermitian metric h, which is convergent by an increasing sequence h_j of C^2-smooth Nakano semi-positive hermitian metrics, is Nakano semi-positive.

- L. Lempert. Modules of square integrable holomorphic germs. in "Trends in Mathematics", 311–333, 2017.

✓ well-known : h_j Nakano semi-positive $\Rightarrow$ h_j is L^2 optimal ;
✓ consequently the limit metric h is also L^2 optimal ;
✓ Then by Theorem DNWZ, h is Nakano semi-positive in the usual sense.

Guan-Zhou's solution of the strong openness conjecture may be reformulated for the pseff line bundles (L, h).

Let (L, h) be a pseudoeffective line bundle, i.e., the curvature current of h is semi-positive, then the associated multiplier ideal sheaf $\mathcal{I}(h)$ satisfies the strong openness. i.e., let a sequence of metrics h_j decreasingly converge to h, then $\bigcup_j \mathcal{I}(h_j) = \mathcal{I}(h)$

It's natural to ask if the multiplier submodule sheaves for the vector bundles satisfy the analogue properties.

Let X be a complex manifold, E be a holomorphic vector bundle of rank r over X.

(1) h is called a *singular Finsler metric* on E over X if $|\cdot|_{h(z)} : E_z \longrightarrow [0, +\infty]$ satisfies that $|c\xi|_{h(z)} = |c||\xi|_{h(z)}$ for any $c \in \mathbb{C}$ and $\xi \in E_z$ and $|F(z)|_{h(z)}$ is measurable in z for any local measurable section F and both sets

$$Z := \{x \in X \mid \exists\, 0 \neq \xi \in E_x \text{ s.t. } |\xi|_{h(x)} = 0\}$$
$$P := \{x \in X \mid \exists\, 0 \neq \xi \in E_x \text{ s.t. } |\xi|_{h(x)} = +\infty\}$$

are of measure 0;

(2) h is called a *singular hermitian metric* on E over X if $\xi \in E_z \longmapsto (\langle \xi, \xi \rangle_{h(z)})^{1/2}$ is a singular Finsler metric such that $h(z)$ is a measurable non-negative hermitian metric on $X \setminus P$, i.e. $\langle \cdot, \cdot \rangle_{h(z)} : E_z \times E_z \longrightarrow \mathbb{C}$ is hermitian semi-positive defined for each $z \in X \setminus P$.

The following fact is well-known :
Let h be a smooth hermitian metric on E. Then the following are equivalent :
(1) h is smooth Griffiths semi-negative.
(2) $|u|_h$ is plurisubharmonic for any local holomorphic section u of E.
(3) $\log |u|_h$ is plurisubharmonic for any local holomorphic section u of E.
(4) the dual metric h^* on E^* is smooth Griffiths semi-positive.

Let E be a holomorphic vector bundle over a complex manifold X.
(1) A metric h on E is singular Griffiths semi-negative if h is a singular hermitian metric and $|u|_h$ is plurisubharmonic for any local holomorphic section u of E.
(2) A singular hermitian metric h is Griffiths semi-positive if h^* is singular Griffiths semi-negative on E^*.

The above DNWZ theorem on characterization of Nakano semipositivity naturally lead to define Nakano semi-positivity for singular hermitian metrics on holomorphic vector bundles (see Inayama's paper)

▶ We say that a singular hermitian metric h is singular Nakano semi-positive if
(1) h is singular Griffiths semi-positive;
(2) h is L^2 optimal

Singular version of Lempert's problem :

Proposition

Let h be a singular Griffiths semi-positive metric. Let $\{h_j\}$ be a sequence of singular Nakano semi-positive metrics. Assume that $\{h_j\}$ is bounded below by a continuous hermitian metric and $h_j \leq h$ converges to h. Then h is also singular Nakano semi-positive.

For any Stein coordinate U such that $E|_U$ is trivial, any Kähler form ω_U on U and any $\psi \in \mathrm{Spsh}(U) \cap C^\infty(U)$ such that for any $\overline{\partial}$-closed $f \in L^2_{(n,1)}(U, \omega_U, E|_U, he^{-\psi})$ and

$$\int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U,h} e^{-\psi} dV_{\omega_U} < +\infty,$$

we have

$$\int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U,h_j} e^{-\psi} dV_{\omega_U} \leq \int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U,h} e^{-\psi} dV_{\omega_U} < +\infty.$$

Then there exists $u_j \in L^2_{(n,0)}(U, \omega_U, E|_U, h_j e^{-\psi})$ such that $\bar{\partial} u_j = f$ and

$$\int_U |u_j|^2_{\omega_U, h_j} e^{-\psi} dV_{\omega_U} \leq \int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U, h_j} e^{-\psi} dV_{\omega_U}$$

$$\leq \int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U, h} e^{-\psi} dV_{\omega_U}$$

Since $\{h_j\}$ is bounded below by a continuous hermitian metric, we get that $\{u_j\}$ is bounded in $L^2_{\text{loc}}(U)$, and especially $\{u_j - u_1\}$ is bounded in $L^2_{\text{loc}}(U)$.

Then there exists a subsequence of $\{u_j - u_1\}$ compactly converging to $u - u_1$ on U since $\overline{\partial}(u_j - u_1) = 0$.

So we obtain $\overline{\partial}u = f$ and

$$\int_U |u|^2_{\omega_U,h} e^{-\psi} dV_{\omega_U} \leq \int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U,h} e^{-\psi} dV_{\omega_U}.$$

- The multiplier submodule sheaf $\mathcal{E}(h)$ of $\mathcal{O}(E)$ associated to a singular hermitian metric h on E is the sheaf of the germs of $s_x \in \mathcal{O}(E)_x$ such that $|s_x|_h^2$ is integrable in some neighborhood of x.

When E is a line bundle, the above concept is just multiplier ideal sheaf

Theorem (Hosono and Inayama) : The multiplier submodule sheaf $\mathcal{E}(h)$ associated to a singular Nakano semi-positive hermitian metric h is coherent.

Theorem(Liu-Yang-Zhou) The multiplier submodule sheaf $\mathcal{E}(h)$ satisfies strong openness and stability.

Strong openness : Let $(E,h) \geq_{\text{Nak}} 0$ and $(E,h_j) \geq_{\text{Grif}} 0$ for $j \geq 1$. If $h_j \geq h$ converges to h, then $\sum_j \mathcal{E}(h_j) = \mathcal{E}(h)$. Especially, $\mathcal{E}_+(h) = \mathcal{E}(h)$.

Stability : let $\{F_j\} \subset H^0(D,E)$ satisfy that $(F_j, o) \in \mathcal{E}(h)_o$ and F_j compactly converge to $F \in H^0(D,E)$. Then for any $p \in (0, 1 + c_o^F(h))$, $|F_j|^2_{h_j}$ converges to $|F|^2_h$ in L^p in some neighborhood of o.

- $$\mathcal{E}_+(h) = \bigcup_{\beta>0} \mathcal{E}(h(\det h)^\beta).$$
- the complex singularity index $c_o^F(h)$ of h at o respect to F :

$$c_o^F(h) := \sup\{\beta \geq 0 \ : \ |F|^2_h(\det h)^\beta \text{ is integrable near } o\}.$$

"The following very general statement was recently obtained by X. Zhou-L. Zhu as the culmination of many previous works : Ohsawa-Takegoshi, Ohsawa, ..., D, Cao-D-Matsumura (2017)."
cited from Demailly's slides 2021

Let (X,ω) be a holomorphically convex Kähler manifold, ψ be an L^1_{loc} function on X which is locally bounded above, and (L,h) be a singular hermitian line bundle over X.

Assume that $\alpha > 0$ is a positive continuous function on X, and that the following two inequalities hold on X in the sense of currents :
$(i) \quad \sqrt{-1}\Theta_{L,h} + \sqrt{-1}\partial\bar{\partial}\psi \geq 0,$
$(ii) \quad \sqrt{-1}\Theta_{L,h} + (1+\alpha)\sqrt{-1}\partial\bar{\partial}\psi \geq 0.$

Theorem(**Cao-Demailly-Matsumura, Zhou-Zhu**)
Then the homomorphism induced by the natural inclusion $\mathcal{I}(he^{-\psi}) \longrightarrow \mathcal{I}(h)$,

$$H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi})\big) \longrightarrow H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

is injective for every $q \geq 0$.

In other words, the homomorphism induced by the natural sheaf surjection $\mathcal{I}(h) \longrightarrow \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$,

$$H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

$$\longrightarrow H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})\big)$$

is surjective for every $q \geq 0$.

✓ The special case when ψ is a quasi-psh function with neat analytic singularities was proved by Cao-Demailly-Matsumura.
✓ The general case when ψ is a quasi-psh function with arbitrary singularities was posed as a question by Cao-Demailly-Matsumura.

• J.Y. Cao, J.-P. Demailly, and S. Matsumura, *A general extension theorem for cohomology classes on non reduced analytic subspaces*, Sci. China Math. **60** (2017), no. 6, 949–962.
• X.Y. Zhou and L.F. Zhu, *Extension of cohomology classes and holomorphic sections defined on subvarieties*, J. Algebraic Geom. 2022, see also arXiv :1909.08822v1.

Theorem(Zhou-Zhu (JAG 2022))
Let X be a holomorphically convex Kähler manifold. Let (F, h_F) and (G, h_G) be two singular hermitian line bundles over X.
Assume that the following two inequalities hold on X in the sense of currents :
(i) $\quad \sqrt{-1}\Theta_{F,h_F} \geq 0,$
(ii) $\quad \sqrt{-1}\Theta_{F,h_F} \geq b\sqrt{-1}\Theta_{G,h_G}$ for some $b \in (0, +\infty)$.

Then, for a non-zero global holomorphic section s of G satisfying $\sup_{\Omega} |s|_{h_G} < +\infty$ for every $\Omega \subset\subset X$,
the following map β induced by the tensor product with s

$$H^q\big(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F)\big)$$

$$\xrightarrow{\beta} H^q\big(X, \mathcal{O}_X(K_X \otimes F \otimes G) \otimes \mathcal{I}(h_F h_G)\big)$$

is injective for every $q \geq 0$.

✓ the above injectivity theorem on holomorphically convex Kähler manifolds is an application of the above extension theorem of cohomology classes,

✓ the above injectivity theorem unifies and generalizes several well-known injectivity theorems obtained by Matsumura 2014, Fujino-Matsumura 2016, Matsumura 2016, Gongyo-Matsumura 2017, Matsumura 2018.

Thank You!