On Stein neighborhoods of certain Stein domains

— Remarks on the regularity of the boundary

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1. Introduction

1986 Kyoto Trento Wuppertal

L^2 extension theorem

$$\underset{\psi\mathrm{cvx}}{\Omega} \subset \mathbb{C}^n, \quad \Omega' = \Omega \cap \{z_n = 0\}$$

$$\sup_{\Omega} |z_n| < 1, \quad \varphi \in PSH(\Omega)$$

$$\Rightarrow \forall f \in \mathcal{O}(\Omega') \; \exists \tilde{f} \in \mathcal{O}(\Omega) \text{ s.t.}$$

$$\begin{cases} \tilde{f}|_{\Omega'} = f \\ \int_{\Omega} e^{-\varphi} |\tilde{f}|^2 \leq 1620\pi \int_{\Omega'} e^{-\varphi} |f|^2 \end{cases}$$

2012

$1620 \to 1$

Błocki, Guan-Zhou

2004 Maitani-Yamaguchi,

2006 Berndtsson

2014 Berndtsson-Lempert

1992 Nagoya

Bergman kernel of hyperconvex domains

$\underset{\psi\text{cvx}}{\Omega \subset\subset} \mathbb{C}^n \quad \exists \psi \in PSH(\Omega)_- \text{ s.t.}$

$\{\psi < c\} \subset\subset \Omega$ for $\forall c < 0$

$\Rightarrow \lim_{z \to \partial\Omega} B_\Omega(z,z) = \infty$.

1993 Cambridge (USA)

Suita conjecture

$$\pi B_X(z,z) \geq c_\beta(z)^2$$

for $\forall$ Riemann surface X, where

$c_\beta := e^{-\gamma}$ for the Robin constant γ

(Dirichlet Pbm $\sim$ Levi Pbm)

Partial Ans to SC.

$$750\pi B_X(z,z) \geq c_\beta(z)^2$$

$$750 \rightarrow 512 \rightarrow 6 \rightarrow 2$$

Berndtsson Błocki

$$2011 \quad \longrightarrow 1.954 \longrightarrow 1 \quad 2012$$

Guan-Zhou-Zhu Błocki, Guan-Zhou

2. Results on hyperconvex domains

1. L^2 cohomology

Donnelly - Fefferman '83

Gromov '91

$$\dim H^{p,q}_{(2)} = \begin{cases} 0 & p+q \neq n \\ \infty & p+q = n \end{cases}$$

2. $\partial\Omega \in C^{\alpha}$, $\Omega \underset{\psi cvx}{\subset\subset} \mathbb{C}^n$

$\partial\Omega \in Lip \Rightarrow hcvx$ (Demailly '87)

$\partial\Omega \in C^{\alpha}$ $(\alpha > 0) \Rightarrow hcvx$

B.Y. Chen 2021

$\Omega \underset{\psi cvx}{\subset} \mathbb{CP}^n$

$\partial\Omega \in Lip \Rightarrow hcvx$ (Harrington 2017)

$\partial\Omega \in C^{\alpha}$, $\underset{\psi cvx}{\Omega} \subset \mathbb{CP}^n$

$\Rightarrow$ hcvx ?

Neighborhoods of Stein submanifolds

'76 Rossi (Fornaess)

$\underset{s\psi c}{\Omega} \Subset M \underset{\text{cx submfd}}{\subset} N$, N Stein

$\Rightarrow \exists \underset{s\psi c}{\tilde{\Omega}} \subset N$ s.t. $\tilde{\Omega} \cap M = \Omega$

'76/'77 Siu

$\underset{\text{Stein submfd}}{M \subset N} \Rightarrow \exists \underset{\text{Stein nbd}}{U \supset M}$

'62 Grauert $M \Subset M \underset{\text{cx submfd}}{\subset} N$

$$0 \to T_M \to T_N|_M \to N_{M/N} \to 0$$

$N_{M/N} < 0 \Rightarrow \exists \underset{\text{S}\psi\text{C nbd}}{U \supset M}$

'63 a generalization

3. Levi flat Stein domains

Example

$$M = \hat{\mathbb{C}} \times (\mathbb{C}^*/\mathbb{Z})$$

$$\supset \{(\zeta,[z]) \mid \mathrm{Re}(\zeta e^{2\pi i \log|z|}) > 0\} = \Omega_0$$

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \\ \cup & & \cup \\ z & \longmapsto & e^m z \end{array} \qquad m \in \mathbb{Z}$$

$$\Rightarrow \quad \partial\Omega_0 \in C^\omega \quad (\text{Levi flat})$$

$\Omega_0 \simeq$  $\times \mathbb{C}^*$

Weighted Bergman spaces for various special Levi flat domains are calculated by M. Adachi

4. A class of Levi flat Stein domains

R: cpt Riemann surf. $g \geq 2 = \mathbb{D}/\Gamma$

$\rho : \pi_1(R) \longrightarrow PSL(2,\mathbb{R}) \simeq \mathrm{Aut}\,\mathbb{D}$

$$\begin{array}{l} \Omega_\rho := R \times_\rho \mathbb{D} := \mathbb{D} \times \mathbb{D}/\sim \\ \downarrow \qquad (x,\zeta) \sim (\gamma(x), \rho(\gamma)(\zeta)) \quad \forall \gamma \in \Gamma \\ R \end{array}$$

$\Rightarrow \quad \Omega_\rho \subset R \times_\rho \hat{\mathbb{C}} = M_\rho$

$\partial\Omega_\rho \in C^\omega$ Levi flat

Ω_ρ is Stein $\Leftrightarrow$ $\Omega_\rho \to R$ has no hol. sect.

$\exists$ Other classes

i) Nemirovski type

ii) $\exists$ Levi flat Stein domains in Hirzebruch-Inoue surfaces

5. Local and global

Kerzman and Rosay 1981

$\underset{\text{loc. hcvx}}{\Omega \Subset \mathbb{C}^n} \Rightarrow$ hcvx

1998 Diederich – O′

$\exists \; \underset{\text{loc. hcvx}}{\Omega \Subset \mathbb{C}\mathbb{P}^5}$ s.t.

Ω is not hcvx.

<u>Thm 1</u>

$\rho = \rho_\alpha$ for some $\alpha : R \xrightarrow{\text{homeo}} R' \not\ni R$

$\Rightarrow \Omega_{\rho_\alpha}$ has a Stein nbd w. Hölder cont. bdry in M_{ρ_α}

<u>Thm 2</u> Ω_0 has a Stein nbd w. log-Hölder bdry.