Inspiration!

CR-geometry $\longleftrightarrow$ Differential Equations

$$\boxed{M \longleftrightarrow E(M)}$$

analytic CR mfd — integrable system of diff eqs

Applications in the direction from Differential Equations (DS) to Complex Analysis (CR) are plentiful!

($M \longrightarrow E(M)$)

CR geometry	Dynamical Systems
Degenerate hypersurfaces	Singular meromorphic ODEs/PDEs
Infinite type hypersurfaces in C^2	Saddle-node vector fields in C^2
Degeneracy locus	Singular locus of ODEs/PDEs
Segre varieties	Graphs of solutions
CR-maps	Point transformations of differential equations
Infinitesimal automorphisms	Lie symmetries
Fuchsian type hypersurfaces	Fuchsian differential equations
Stokes phenomenon for CR-maps	Stokes phenomenon for DS

Holomorphic extension of maps to degeneracy locus	Meromorphic extension of solutions to singular locus
...	...

Today I will talk about the other direction:

Complex Analysis (CA) to Differential Equations

$$\boxed{E \dashrightarrow M}$$

non-analytic

Present solutions for two problems, both concerning second order ODEs

$$\boxed{y'' = F(x, y, y')}$$

C^ω, C^∞, $C^k, k \geq 1$

* Arnold, 1960's: classify such ODEs locally by means of a complete normal form

$$E = \{y'' = F(x, y, y')\} \rightsquigarrow$$

$$E_N = \{ y'' = N(x, y, y') \}$$

(Reminder: $\{ y' = F(x,y) \} \to \{ y' = 0 \}$)

Group: $(x,y) \xrightarrow[(\mathbb{C}^2,0)]{(\mathbb{R}^2,0)} (f(x,y), g(x,y))$

*Problem (Ghys): Prove Cartan-Tresse's flattening theorem for ODEs in low regularity.

? When can an ODE $\{ y'' = F(x,y,y') \}$ be flattened, i.e. mapped into $\{ y'' = 0 \}$?

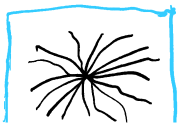
Miracle: the property of an ODE to be cubic in y' is invarinat under diffeos!

So, only ODEs of the kind

$$y'' = A(x,y) + B(x,y) y' + C(x,y)(y')^2 + D(x,y)(y')^3$$

have the potential to be flattened.

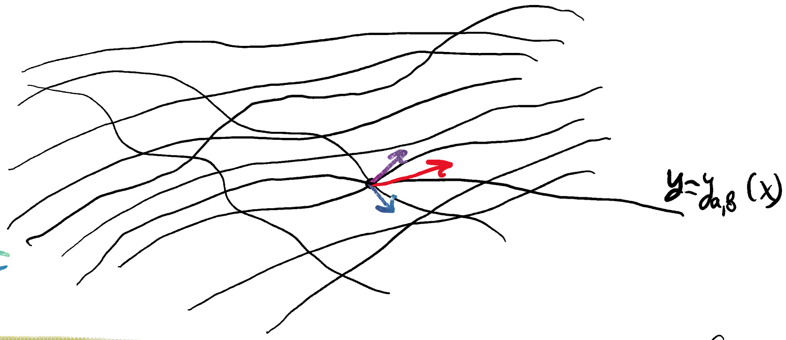
Note: geometrically, cubic ODEs precisely those for which the



Note: geometrically, cubic ODEs are precisely those for which the „pencil" of solutions through a pt forms a foliation

solutions: a 2-parameter family $\{y_{a,\beta}(x)\}_{a,\beta\in\mathbb{R}}$

phase portrait of a 2d order ODE



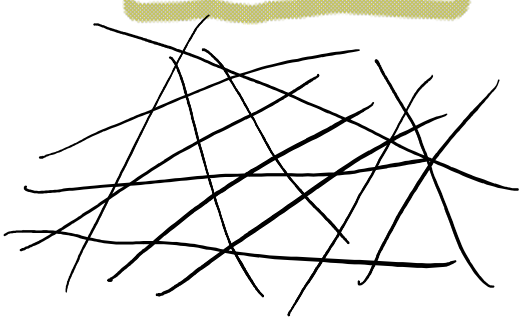
$$\boxed{y=\Theta(x,a,\beta)}$$

could be
$y(0)=\beta$
$y'(0)=a$

parameterized family of solutions

Flattening: transforming into the „flat" phase portrait

$$\boxed{y=ax+\beta}$$



$$\beta=\Theta^*(a,x,y)$$

$$\downarrow$$

$$\boxed{\beta''=F^*(a,\beta,\beta')}$$

dual ODE defined up to a local diffeo

For the flat ODE, $F^*=0 \Rightarrow$

F^* is cubic in β' for a flattenable ODE!

F^* is cubic in δ' for a flattenable ODE!

Theorem: for $F \in C^\infty$ or $F \in C^\omega$, the ODE $E = \{y'' = F(x,y,y')\}$ can be flattened $\Longleftrightarrow$ E and its dual E^* are both cubic in the derivative. (Tresse, ≈ 1900; E. Cartan, 1920's)

Tresse's proof: differential invar-s

Cartan's proof: moving frame

can be extended to $F \in C^k$, $k \geq 4$, but not below.

Problem: extend to the regularity $\boxed{F \in C^1}$

??Why C^4??

Achieve by applying the simingly unrelated Complex Analysis!

Normal form problem

* Tresse, 1900's: solution of the equiv. prob. for ODEs via differential invariants (starting from the above $F_{y'y'y'y'}$)
* E. Cartan, 1920's: solution of the equiv. prob. for ODEs by means of moving frames

prob. for ODEs by means of moving frames

* Arnold, 1960's: a partial normal form (the problem remained unsolved!)

We have found two distinct solutions.

$\boxed{\text{Solution 1}}$ (works for $F \in C^\infty$ or C^ω)

Based on similarity with the case of Levi-nondegenerate hypersurfaces in $\mathbb{C}^2$

– From the ODE $\{y'' = F(x, y, y')\}$ switch to its (param-d) family of solutions

$$\{y = \Theta(x, a, b)\} = M \subset \mathbb{R}^2_{(x,y)} \times \mathbb{R}^2_{(a,b)}$$

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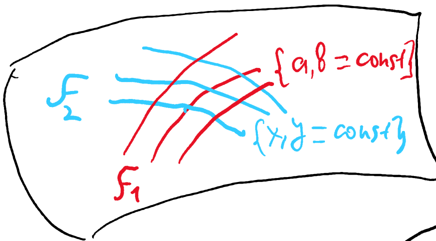

$\mathcal{F}_1$ – solutions

$\mathcal{F}_2$ – dual solutions

$[\mathcal{D}_1, \mathcal{D}_2] \notin \text{span}\{\mathcal{D}_1, \mathcal{D}_2\}$

"Levi-form"

$\boxed{y = ax + b} + O(2)$ – local model

extreme similarity with the complexification M^c of a real-anal hypersurface; $\mathcal{F}_1$ – Segre, $\mathcal{F}_2$ – dual Segre

transformations $(x, y) \to (f(x,y), g(x,y))$ of E $\longleftrightarrow$ "product" transformations $(x, y, a, b) \to (f(x,y), g(x,y), \lambda(a,b), \mu(a,b))$ of M

Classifying the M's is a development of the Chern-Moser procedure:

– assign weight coming from the model:

$$\boxed{[x] = [a] = 1, \quad [y] = [b] = 2}$$

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$$M: \quad y = b + ax + \sum_{j=3}^{\infty} \Theta_j(x, a, b)$$

$$\tilde{M}: \quad y = b + ax + \sum_{j=3}^{\infty} \tilde{\Theta}_j(x, a, b)$$

$$H: (M, 0) \to (\tilde{M}, 0), \quad H = H_s \circ \Psi,$$

$$\Psi \in PGL_0^{(x,y)}(2, \mathbb{R}) \times PGL_0^{(a,b)}(2, \mathbb{R}),$$

related

with 3 specific Taylor coef. vanishing

$$H_s = (f(x,y), g(x,y), \lambda(a,b), \mu(a,b))$$

Taylor expansion

$$= \left(x + \sum_{j=2}^{\infty} f_j(x,y),\ y + \sum_{j=3}^{\infty} g_j(x,y),\ a + \sum_{j=2}^{\infty} \lambda_j(a,b),\ b + \sum_{j=3} \mu_j(a,b) \right)$$

Then $\boxed{H(M) \subset \tilde{M}}$ reads as a series of equations:

functional eqn for (f,g)

$$L(f_{j-1}, g_j, \lambda_{j-1}, \mu_j) = \tilde{\Theta}_j - \Theta_j + \ldots \quad \Big| \ j = 3, 4, \ldots$$

where $L(f,g,\lambda,\mu) := g(x,y) - a f(x,y) - x\lambda(a,b) - \mu(a,b) \Big|_{y=b+ax}$

<u>Proposition</u>: Every formal solution manifold M as above can be brought to a normal form:

$$y = b + ax + \sum_{k,l \geq 2} N_{kl}(b)\, x^k a^l, \quad N_{22} = N_{23} = N_{32} = N_{33} = 0.$$

A normalizing diffeo is unique, up to a choice of a projective transform $\Psi_1 \in PGL(2, \mathbb{R})$ preserving $(0,0,0) \in J^1(\mathbb{R}, \mathbb{R})$.

Accordingly, a formal ODE $E = \{y'' = F(x,y,y')\}$ can be brought to a normal form $E_N = \{y'' = N(x,y,y')\}$,

$$\boxed{N(x,y,y') = O(x^2 y'^2) + O(y'^4)}$$

...

$$\boxed{N(x,y,y') = O(x^2y'^2) + O(y'^4)}$$

(with a similar uniqueness assertion).

The "bundle" of formal normal forms at all points $p \in M$ $\in C^\infty$ allows to define a canonical direction field on a bundle $\pi: X \to M$;

Projections $\pi(\gamma)$ define canonical curves in M called chains (= curves that get straightened in the normal form $\{x=a=0, y=b\}$)

$\exists!$ chain through $p \in M$ in $\forall$ direction transverse to the canonical foliations

Using the chains and the formal normal form, we finally prove

Theorem 1 (K–Zaitsev, 2018, JDCS)

(1) Every (C^∞ or C^ω) manifold of solutions M as above can be brought near $0 \in M$ to a normal form

$$\boxed{\{y = b + ax + O(x^4a^2) + O(x^2a^4)\}}$$

Approximation by the "flat" phase portrait to order 5

(2) Every (C^∞ or C^ω) ODE $\{y'' = F(x,y,y')\}$ can be brought near $0 \in J^1(\mathbb{R},\mathbb{R})$ to a normal form

$$\boxed{\{y'' = N(x,y,y')\}, \quad N(x,y,y') = O(x^2y'^2) + O(y'^4)}$$

Approximation to order 3 by "flat" ODE

In both cases, the normal form is unique, up to a choice of $\varphi_1 \in PGL(2,\mathbb{R})$ preserving $0 \in J^1(\mathbb{R},\mathbb{R})$.

The normal form and the normalizing diffeo are respectively C^∞ or C^ω.

Proof: straightening a chain + performing a sequence of explicit transformations

Proof: straightening a chain + performing a sequence of explicit transformations + using the chain property / formal normal form.

Remark: an inspection of the proof allows to extend it to the regularity $F \in C^4$ ($\sim \theta \in C^6$)

Ex. of application: analytic regularizability ($E \overset{diff}{\sim}$ analytic ODE $\Leftrightarrow E_N$ is analytic)

Solution 2 (works for C^ω)

$$E = \{y'' = F(x, y, y')\}$$

We make use of $\mathcal{J}^k(\mathbb{R}, \mathbb{R})$ - bundles of k-jets of solutions (= curves in $\mathbb{R}^2$)

$\mathcal{J}^1(\mathbb{R}, \mathbb{R}) = \mathbb{P}T^*\mathbb{R}^2$,

Domains of 2d order ODEs = domains in $\mathcal{J}^1$

ODE $E = \{y'' = F(x, y, y')\}$ - a hypers. in $\mathcal{J}^2$

$\mathcal{J}^k$ - bundle over $\mathcal{J}^{k-1}$

(local) diffeos $H: \mathbb{R}^2 \to \mathbb{R}^2$ extend to $H^{(k)}: \mathcal{J}^k \to \mathcal{J}^k$

So, classifying 2d order ODEs under $\mathrm{Diff}(\mathbb{R}^2, 0) \simeq$
$\simeq$ classifying hypersurfaces in $\mathcal{J}^2(\mathbb{R}, \mathbb{R}) \sim \mathbb{R}^4$ under the special group $G := j^2\mathrm{Diff}(\mathbb{R}^2, 0) \subset \mathrm{Diff}(\mathbb{R}^4, 0)$

$$H^{(2)}(x, y) = \left(f(x, y),\ g(x, y),\ g^1(x, y, y'),\ g^2(x, y, y', y'')\right)$$

$$g^1(x, y) = \frac{g_x + y' g_y}{f_x + y' f_y}$$

$$g^2(x, y) = P(x, y, y') + y'' \cdot Q(x, y, y')$$

P, Q - rational expressions in y' and $j^2 f$, $j^2 g$

$g^2(x,y) = P(x,y,y') + y'' \cdot Q(x,y,y')$ | y' and $j^2 f$, $j^2 g$

This leads to the following transformation rule for ODEs $E = \{y'' = F(x,y,y')\}$, $\tilde{E} = \{y'' = \tilde{F}(x,y,y')\}$:

$$F(x,y,y') = \frac{1}{J}\Big((f_x + y' f_y)^3 \cdot \tilde{F}\left(f(x,y), g(x,y), \frac{g_x + y' g_y}{f_x + y' f_y}\right) + I_0 + I_1 y' + I_2 (y')^2 + I_3 (y')^3\Big),$$

$J := f_x g_y - f_y g_x$ is the Jacobian determinant of the transformation and

$$I_0 = g_x f_{xx} - f_x g_{xx}$$
$$I_1 = g_y f_{xx} - f_y g_{xx} - 2 f_x g_{xy} + 2 g_x f_{xy}$$
$$I_2 = g_x f_{yy} - f_x g_{yy} - 2 f_y g_{xy} + 2 g_y f_{xy}$$
$$I_3 = g_y f_{yy} - f_y g_{yy}.$$

It's convenient to treat $\tilde{F}$ as source, F as target.

Expand $F(x,y,y') = \sum_{j \geq 0} F_j(x,y)(y')^j$ $\quad (F \in C^\omega)$

Pose the conditions: $\boxed{F_0(x,y) = F_1(x,y) = 0}$

Amounts to solving a Cauchy problem:

$$f_{xx} = U(f, g, f_x, f_y, g_x, g_y, f_{xy}, f_{yy}, g_{xy}, g_{yy})$$
$$g_{xx} = V(\text{———} \; // \; \text{———})$$

significantly use the analyticity

To uniquely determine $H = (f,g)$, the Cauchy data $[f_0(y) = f(0,y),\ g_0(y) = g(0,y),\ f_1(y) = f_x(0,y),\ g_1(y) = g_x(0,y)]$ is missing.

We pose: $\boxed{F_2(0,y) = F_3(0,y) = \frac{\partial F_2}{\partial x}(0,y) = \frac{\partial F_3}{\partial x}(0,y) = 0}$

Amounts to a system of ODEs:

$$Y'' = W(Y, Y'), \quad Y := (f_0, g_0, f_1, g_1)$$

Initial data $\simeq$ parameters of a projective map.

This proves Theorem 1 in the C^ω-case

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The idea goes back to Ebenfelt-K.-Lamel, TAMS, 2022 (normal form for ∞ type hypersurfaces).

Remark: Based on the same idea, one can much easier (than originally) deduct the Chern-Moser analytic normal form!

Application of the normal form:

Ghys Problem: Let $E=\{y''=F(x,y,y')\}$, $F\in C^1$. Let both E, E^* have resp. F, F^* cubic in y'. Prove that E can be flattened.

$\boxed{\text{Achieved by combining both methods!}}$

Key claim: a cubic ODE can be still brought to a normal form in „low regularity" (!)

Step 1: „chain equation" $Y''=W(Y,Y')$

„W is C^{k-1} if F is C^k" follows from the cubicity! so, even for $k=1$ we can find chains by using Peano Theorem.

Step 2: appealing now to the family of solutions $\{y=\theta(x,a,b)\}$ we can, after straightening the chains, achieve the conditions:

$\theta(0,a,b)=\theta(x,0,b)=b$, $\theta_x(0,a,b)=a$, $\theta_a(x,0,b)=x$

Then E becomes: $\boxed{y''=P(x,y)(y')^2+Q(x,y)(y')^3}$

The chain cond.: $\boxed{P(0,y)=Q(0,y)=\frac{\partial P}{\partial x}(0,y)=\frac{\partial Q}{\partial x}(0,y)=0}$

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Using the dual: $b'' = F^*(a, b, b')$

$y = \Theta(x, a, b) \leftrightarrow b = \Theta^*(a, x, y)$

Claim (Tresse, et al) $\dfrac{\partial^4 F^*}{(\partial b')^4} = k(\Theta) \cdot \mathcal{D}(F)$,

where $k(\Theta)$ is a nonvanishing factor, and

$\mathcal{D}(F) := -F_{xxy'y'} + 4F_{xyy'} - 6F_{yy} + F_{y'} \cdot F_{xy'y'} -$
$- 4F_{y'}F_{yy'} + 3F_y F_{y'y'}$

So, $\boxed{E^* \text{ is cubic in } b' \iff \mathcal{D}(F) \equiv 0}$

For F which is cubic and in normal form, the eqn $\boxed{\mathcal{D}(F) = 0}$ can be solved directly, and gives: $\boxed{P = Q = 0}$

So, $\boxed{E = \{y'' = 0\}}$ - already flattened.

$\boxed{\text{Theorem 2}}$ (K.-Zaitsev, 2024)

Let $E = \{y'' = F(x, y, y')\}$ satisfies $F \in C^2$
(or the weaker condition $F \in C^1$, $\Theta \in C^2$).
Assume E, E^* are cubic in y', b' resp.
Then E is flattenable (with the diffeo $H \in C^2$).

THANK YOU FOR YOUR ATTENTION!

YOUR ATTENTION!

Wuppertal

Friday, October 18, 2024 9:42 AM

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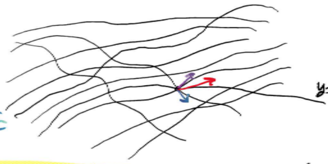
phase portrait of a 2d order ODE



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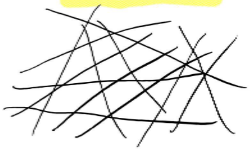
Flattening: transforming into the "flat" phase portrait $\boxed{y = ax+b}$



$b = \Theta^*(a, x, y)$

$\downarrow$

$\boxed{b'' = F^*(a, b, b')}$ — dual ODE — defined up to defined up to a local diffeo

For the flat ODE, $F^* = 0 \Rightarrow$
F^* is cubic in y' for a flattenable ODE!

Theorem: for $F \in C^\infty$ or $F \in C^\omega$, the ODE $E = \{y'' = F(x, y, y')\}$ can be flattened $\Leftrightarrow$ E and its dual E^* are both cubic in the derivative.
(Tresse, ≈ 1900; E. Cartan, 1920's)

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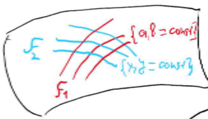

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$$\widetilde{M}: \quad y = b + ax + \sum_{j=3}^{\infty} \widetilde{\Theta}_j(x,a,b)$$

$$H: (M,0) \to (\widetilde{M},0), \quad H = H_s \circ \Psi,$$

$$\Psi \in PGL_0^{(x,y)}(2,\mathbb{R}) \underset{\text{related}}{\times} PGL_0^{(a,b)}(2,\mathbb{R}),$$

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$$H_s = (f(x,y), g(x,y), \lambda(a,b), \mu(a,b)) \overset{\text{Taylor}}{\underset{\text{expansion}}{\sim}}$$

$$= \left(x + \sum_{j=2}^{\infty} f_j(x,y),\ y + \sum_{j=3}^{\infty} g_j(x,y),\ a + \sum_{j=2}^{\infty} \lambda_j(a,b),\ b + \sum_{j=3}^{\infty} \mu_j(a,b) \right)$$

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└ functional eqn for (f,g)

$$L(f_{j-1}, g_j, \lambda_{j-1}, \mu_j) = \widetilde{\Theta}_j - \Theta_j + \ldots \quad \Big|_{j=3,4,\ldots}$$

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Using the chains and the formal normal form, we finally prove

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$$\boxed{\{y = b + ax + O(x^4 a^2) + O(x^2 a^4)\}}$$

Approximation by the „flat" phase portrait to order 5

(2) Every (C^∞ or C^ω) ODE $\{y'' = F(x,y,y')\}$ can be brought near $0 \in J^1(\mathbb{R},\mathbb{R})$ to a normal form

$$\boxed{\{y'' = N(x,y,y')\} \quad N(x,y,y') = O(x^2 y'^2) + O(y'^4)}$$

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$\boxed{\{y'' = N(x,y,y')\},\ N(x,y,y') = O(x^2 y'^2) + O(y'^4)}$

Approximation to order 3 by „flat" ODE

In both cases, the normal form is unique, up to a choice of $\varphi_1 \in PGL(2,\mathbb{R})$ preserving $0 \in \mathcal{J}^1(\mathbb{R},\mathbb{R})$.

The normal form and the normalizing diffeo are respectively C^∞ or C^ω.

Proof: straightening a chain + performing a sequence of explicit transformations + using the chain property/formal normal form.

Remark: an inspection of the proof allows to extend it to the regularity $F \in C^4$ ($\sim \theta \in C^6$)

Ex. of application: analytic regularizability ($E \overset{\text{diff}}{\sim}$ analytic ODE $\Leftrightarrow E_N$ is analytic)

Solution 2

(works for C^ω)

$$E = \{y'' = F(x,y,y')\}$$

We make use of $\mathcal{J}^k(\mathbb{R},\mathbb{R})$ - bundles of k-jets of solutions (= curves in $\mathbb{R}^2$)

$\mathcal{J}^1(\mathbb{R},\mathbb{R}) = \mathbb{P}T^*\mathbb{R}^2$,

Domains of 2d order ODEs = domains in $\mathcal{J}^1$

ODE $E = \{y'' = F(x,y,y')\}$ - a hypers. in $\mathcal{J}^2$

$\mathcal{J}^k$ - bundle over $\mathcal{J}^{k-1}$

J^k - bundle over J^{k-1}

(local) diffeos $H: \mathbb{R}^2 \to \mathbb{R}^2$ extend to $H^{(k)}: J^k \to J^k$

So, classifying 2d order ODEs under $\mathrm{Diff}(\mathbb{R}^2,0) \simeq$
$\simeq$ classifying hypersurfaces in $J^2(\mathbb{R},\mathbb{R}) \sim \mathbb{R}^4$ under
the special group $\mathfrak{G} := j^2 \mathrm{Diff}(\mathbb{R}^2,0) \subset \mathrm{Diff}(\mathbb{R}^4,0)$

$$H^{(2)}(x,y) = \left(f(x,y),\ g(x,y),\ g^1(x,y,y'),\ g^2(x,y,y',y'')\right)$$

$$g^1(x,y) = \frac{g_x + y' g_y}{f_x + y' f_y}$$

$$g^2(x,y) = P(x,y,y') + y'' \cdot Q(x,y,y')$$

P, Q - rational expressions in y' and $j^2 f$, $j^2 g$

This leads to the following transformation rule for ODEs $E = \{y'' = F(x,y,y')\}$, $\tilde{E} = \{y'' = \tilde{F}(x,y,y')\}$:

$$F(x,y,y') = \frac{1}{J}\Bigg((f_x + y' f_y)^3 \cdot \tilde{F}\left(f(x,y), g(x,y), \frac{g_x + y' g_y}{f_x + y' f_y}\right) + I_0 + I_1 y' + I_2 (y')^2 + I_3 (y')^3\Bigg),$$

$J := f_x g_y - f_y g_x$ is the Jacobian determinant of the transformation and

$$I_0 = g_x f_{xx} - f_x g_{xx}$$
$$I_1 = g_y f_{xx} - f_y g_{xx} - 2 f_x g_{xy} + 2 g_x f_{xy}$$
$$I_2 = g_x f_{yy} - f_x g_{yy} - 2 f_y g_{xy} + 2 g_y f_{xy}$$
$$I_3 = g_y f_{yy} - f_y g_{yy}.$$

It's convenient to treat $\tilde{F}$ as source, F as target.

Expand $F(x,y,y') = \sum_{j \geq 0} F_j(x,y)(y')^j$ $\quad (F \in C^\omega)$

Pose the conditions: $\boxed{F_0(x,y) = F_1(x,y) = 0}$

Amounts to solving a Cauchy problem:

$$f_{xx} = U(f, g, f_x, f_y, g_x, g_y, f_{xy}, f_{yy}, g_{xy}, g_{yy})$$
$$g_{xx} = V(\text{———} \quad // \quad \text{———})$$

significantly use the analyticity

To uniquely determine $H = (f,g)$, the Cauchy data $[f_0(y) = f(0,y),\ g_0(y) = g(0,y),\ f_1(y) = f_x(0,y),\ g_1(y) = g_x(0,y)]$

data $[f_0(y)=f(0,y),\ g_0(y)=g(0,y),\ f_1(y)=f_x(0,y), g_1(y)=g_x(0,y)]$
is missing.

We pose: $\boxed{F_2(0,y)=F_3(0,y)=\frac{\partial F_2}{\partial x}(0,y)=\frac{\partial F_3}{\partial x}(0,y)=0}$

Amounts to a system of ODEs:

$$Y''=W(Y,Y'),\quad Y:=(f_0,g_0,f_1,g_1)$$

Initial data $\simeq$ parameters of a projective map.

$\boxed{\text{This proves Theorem 1 in the } C^\omega\text{-case}}$

The idea goes back to Ebenfelt-K.-Lamel, TAMS, 2022 (normal form for ∞ type hypersurfaces).

Remark: Based on the same idea, one can much easier (than originally) deduct the Chern-Moser analytic normal form!

Application of the normal form:

Ghys Problem Let $E=\{y''=F(x,y,y')\}$, $F\in C^1$.
Let both E,E^* have resp. F,F^* cubic in y'.
Prove that E can be flattened.

$\boxed{\text{Achieved by combining both methods!}}$

Key claim: a cubic ODE can be still brought to a normal form in „low regularity" (!)

Step 1: chain equation" $Y''=W(Y,Y')$ ~ Follows from the cubicity!

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$\boxed{W \text{ is } C^{k-1} \text{ if } F \text{ is } C^k}$; so, even for $k = 1$ we can find chains by using Peano Theorem.

Step 2: appealing now to the family of solutions $\{y = \theta(x, a, b)\}$ we can, after straightening the chains, achieve the conditions:

$\theta(0, a, b) = \theta(x, 0, b) = b$, $\theta_x(0, a, b) = a$, $\theta_a(x, 0, b) = x$

Then E becomes: $\boxed{y'' = P(x,y)(y')^2 + Q(x,y)(y')^3}$

The chain cond.: $\boxed{P(0,y) = Q(0,y) = \frac{\partial P}{\partial x}(0,y) = \frac{\partial Q}{\partial x}(0,y) = 0}$

So, $\boxed{E \text{ is in normal form}}$

Using the dual: $b'' = F^*(a, b, b')$

$\boxed{y = \theta(x, a, b)} \leftrightarrow b = \theta^*(a, x, y)$

Claim (Tresse, et al) $\frac{\partial^4 F^*}{(\partial b')^4} = k(\theta) \cdot \mathcal{D}(F)$,

where $k(\theta)$ is a nonvanishing factor, and

$\mathcal{D}(F) := -F_{xxy'y'} + 4F_{xyy'} - 6F_{yy} + F_{y'} \cdot F_{xy'y'} - 4F_{y'}F_{yy'} + 3F_y F_{y'y'}$

So, $\boxed{E^* \text{ is cubic in } b' \iff \mathcal{D}(F) \equiv 0}$

For F which is cubic and in normal form, the eqn $\boxed{\mathcal{D}(F) = 0}$ can be solved directly, and gives: $\boxed{P = Q = 0}$

So, $\boxed{E = \{y'' = 0\}}$ - already flattened.

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So, $\boxed{E = \{y'' = 0\}}$ - already flattened.

$\boxed{\text{Theorem 2}}$ (K.-Zaitsev, 2024)

Let $E = \{y'' = F(x, y, y')\}$ satisfies $F \in C^2$ (or the weaker condition $F \in C^1$, $\theta \in C^2$). Assume E, E^* are cubic in y', β' resp. Then E is flattenable (with the diffeo $H \in C^2$).

THANK YOU FOR YOUR ATTENTION!