Hölder continuous solutions of complex Hessian equations

Sławomir Kołodziej (joint work with Cuong Ngoc Nguyen)

Jagiellonian University

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Elementary symmetric polynomials

- To define the hessian equation denote by $S_m(\lambda)$ the elementary symmetric polynomial of degree m in λ:

$$S_m(\lambda) = \sum_{0<j_1<\ldots<j_m\leq n} \lambda_{j_1}\lambda_{j_2}, \ldots \lambda_{j_m}.$$

and consider the "positive cone" Γ_m $(m \leq n)$

$$\Gamma_m = \{\lambda \in \mathbb{R}^n | \ S_1(\lambda) > 0, \ \cdots, \ S_m(\lambda) > 0\}$$

which is convex and the function $S_m^{\frac{1}{m}}$ is concave when restriced to Γ_m (by Gårding).

Elementary symmetric polynomials

- Assume that

$$\lambda_1 \geq \cdots \geq \lambda_m \geq \cdots \geq \lambda_n.$$

Ivochkina's characterisation of the cone Γ_m tells that if $\lambda \in \Gamma_m$, then for $S_{k;i}(\lambda) := S_k(\lambda)_{\lambda_i=0} = \frac{\partial S_{k+1}}{\partial \lambda_i}(\lambda)$

$$S_{k;i_1,\ldots,i_t}(\lambda) > 0$$

for all $\{i_1, ..., i_t\} \subseteq \{1, ..., n\}$, $k + t \leq m$. In particular, if $\lambda \in \Gamma_m$, then at least m of the numbers $\lambda_1, ..., \lambda_n$ are positive.

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- Lin-Trudinger inequality: there exists a constant θ, depending on n and m, such that for any $\lambda \in \Gamma_m$ and $i \geq m$

$$\frac{S_{m;i}(\lambda)}{S_m(\lambda)} \geq \theta$$

  (It does not hold for $i < m$.)

Complex m-Hessian equations in $\mathbb{C}^n$

- The equation is elliptic on the set of m-subharmonic functions (m-sh). A $\mathcal{C}^2(\Omega)$ function u is called m-sh if for any $z \in \Omega$ the Hessian matrix $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(z)$ has eigenvalues in the closure of the cone Γ_m. In other words

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \;\; k = 1, \cdots, \;\; m. \qquad (*)$$

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- For complex dimension 1 we recover the Poisson equation, for top dimension n it is the complex Monge-Ampère equation. The latter is special since this is the only case when we deal only with nonnegative eigenvalues. Nevertheless one can try to adapt the methods working for real Hessian or complex Monge-Ampère equations. In particular pluripotential theory can be extended to m-sh functions.

Complex m-Hessian equations in $\mathbb{C}^n$

- Following the method of Caffarelli-Nirenberg-Spruck for real Hessian equation Vinacua '86, S. Y. Li '06 solved the Dirichlet problem for smooth data and strictly positive right hand side in $(m-1)$-pseudoconvex Ω (that means that Levi form at any point $p \in \partial\Omega$ has its eigenvalues in the cone Γ_{m-1}).

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- The pluripotential approach was introduced by Błocki who defined m-sh function as a subharmonic function satisfying

$$dd^c u \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0$$

for any collection of m-positive $(1,1)$ forms α_j (α is m-positive if $\alpha^j \wedge \beta^{n-j} \geq 0$ for $j = 1, ..., m$).
Dinew, K obtained weak solutions showing that for $q > n/m$, $f \in L^q(\Omega)$ and continuous φ on $\partial\Omega$ there exists $u \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$ satisfying

$$H_m(u) := (dd^c u)^m \wedge \beta^{n-m} = f \beta^n$$

and $u = \varphi$ on $\partial\Omega$.

Hölder continuous solutions

- For $\varphi \in C^{1,1}$ the above solutions are Hölder continuous by C. N. Nguyen (with extra assumption) and Charabati. Here we wish to get the same conclusion for more general class of measures on the RHS of the equation, assuming the existence of a Hölder continuous subsolution.

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- Thus we consider a positive measure μ such that

$$\mu \leq H_m(\varphi),$$

where $\varphi \in SH_m(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ with $0 < \alpha \leq 1$.

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- Thus we consider a positive measure μ such that

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where $\varphi \in SH_m(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ with $0 < \alpha \leq 1$.
- Given the boundary data $\phi \in C^{0,\alpha}(\partial\Omega)$ we wish to solve the Hessian equation

$$u \in SH_m(\Omega) \cap C^0(\bar{\Omega}), \quad H_m(u) = \mu, \quad \text{and } u = \phi \quad \text{on } \partial\Omega.$$

Hölder continuous solutions

Theorem

Let μ be the measure for which there exists a subsolution. Then, the Dirichlet problem above is solvable and there exists $0 < \alpha' \leq 1$ such that

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- This result was proved before by us for a compactly supported measure μ in Ω. Later, Benali-Zeriahi (in a special case) and Charabati - Zeriahi solved the Dirichlet problem for any μ having finite total mass, i.e., $\mu(\Omega) < +\infty$. Now we remove this extra hypothesis. However the Hölder exponent from our last proof is worse than in Charabati - Zeriahi paper.

- In general the complex Hessian measure of a Hölder continuous m-subharmonic function may have unbounded total mass. Let Ω be a strictly m-pseudoconvex domain in $\mathbb{C}^n$ and let ρ be a C^2 strictly m-subharmonic defining function of Ω. Consider the function $-(-\rho)^\alpha$ for some $0 < \alpha < 1$. Then, it is Hölder continuous on $\bar{\Omega}$ and m-subharmonic in Ω. It is easy to see that for any $x \in \partial\Omega$ and U_x a neighborhood of x,

$$\int_{\Omega \cap U_x} H_m(-|\rho|^\alpha) = +\infty.$$

It follows that the measure $\mu = \mathbf{1}_{U_x} \cdot H_m(-|\rho|^\alpha)$ has unbounded mass on Ω and $\mu \leq H_m(-|\rho|^\alpha)$.

About the proof

- Let ρ be a smooth strictly m-sh (in a neighborhood of $\bar{\Omega}$) defining function of Ω, and define for $\varepsilon > 0$ the sets $\Omega_\varepsilon = \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) > \varepsilon\}$, and $D_\varepsilon = \{z \in \Omega : \rho(z) < -\varepsilon\}$.

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- $$\varphi_\varepsilon := \max\left\{\varphi - \varepsilon, \frac{M\rho}{\varepsilon}\right\},$$
  where $M = 1 + \|\varphi\|_{L^\infty(\Omega)}$ is a Hölder continuous subsolution on a bigger domain (we work with zero bdry data).

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Theorem

There exist constants $A_1 > 0$ and $\alpha_0 > 0$ such that for any compact set $K \subset \Omega$

$$\int_K H_m(\varphi) \leq A_1 \left[cap_m(K)\right]^{1+\alpha_0}, \quad \int_K H_m(\varphi_\varepsilon) \leq \frac{CA^{2m}}{\varepsilon^{2m}}\left[cap_m(K)\right]^{1+\alpha_0},$$

with constants depending only $\|\varphi\|^m_{L^\infty(\Omega)}\|\varphi\|^m_{C^{0,\alpha}(\bar{\Omega})}$ and α_0 which is explicitly computable in terms of α, m and n.

About the proof - stability

Proposition

Let u be the solution for μ and let $v \in SH_m \cap L^\infty(\Omega)$ be such that $v = u$ on $\Omega \setminus \Omega_\varepsilon$. Then, there is $0 < \alpha_2 < 1$ such that

$$\sup_\Omega (v - u) \leq \frac{C}{\varepsilon^{2m}} \left(\int_\Omega \max\{v - u, 0\} d\mu \right)^{\alpha_2},$$

where $C > 0$ is a uniform constant independent of ε.

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Let u be the solution and let $v \in SH_m \cap L^\infty(\Omega)$ be such that $v = u$ on $\Omega \setminus \Omega_\varepsilon$. Then, there is $0 < \alpha_3 < 1$ such that

$$\int_\Omega |v - u| d\mu \leq \frac{C}{\varepsilon^{m+1}} \left(\int_\Omega |v - u| dV_{2n} \right)^{\alpha_3},$$

where $C > 0$ is a uniform constant independent of ε.

About the proof

- For the solution u define for $z \in \Omega_\delta$,

$$\hat{u}_\delta(z) := \frac{1}{\sigma_{2n}\delta^{2n}} \int_{|\zeta|\leq\delta} u(z+\zeta)dV_{2n}(\zeta),$$

where σ_{2n} is the volume of the unit ball. Then, for $\delta > 0$ small,

$$\int_{\Omega_\delta} |\hat{u}_\delta - u| dV_{2n} \leq C\delta.$$

(based on the Jensen formula).

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  (based on the Jensen formula).
- To show Hölder continuity of u we need to show that $\sup_{\Omega_\delta}(u_\delta - u) \leq C\delta^{\alpha'}$ for $u_\delta(z) := \sup_{|\zeta|\leq\delta} u(z+\zeta)$.
  The above integral estimate, stability estimates for some modified regularizations yield that.

- Hessian equations on compact Kähler manifolds

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n, \quad \int_X f\omega^n = \int_X \omega^n$$

(ω a Kähler form on compact manifold X, $f \geq 0$ the given function, $1 < m < n$) The solutions should be (m, ω)-sh and satisfy

$$(\omega + dd^c u)^k \wedge \omega^{n-k} \geq 0, \quad k = 1, ..., m.$$

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- The solution of this equation generalizes Calabi-Yau theorem. In fact the continuity method from Yau's proof can be adapted (not directly) in case of non-negative bisectional curvature. The missing part in the general case are C^2 estimates.

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- Hou-Ma-Wu refined a method of Chou and Wang (real Hessian equations on Riemannian manifolds) reducing the problem to C^1 estimate. They proved the following.

Theorem

If $u \in C^4$ solves the above equation then for an independent C

$$\sup_X ||dd^c u||_\omega \leq C(\sup_X ||\nabla u||^2 + 1).$$

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- C^1 estimate was proven by Dinew and K. and it was based on

Theorem

Any bounded, maximal m-sh function with bounded gradient in $\mathbb{C}^n$ is constant. Therefore the Hessian equation on a compact Kähler manifold is solvable for smooth, positive f.

Also using similar argument as in M-A case we proved that there exist weak continuous solutions of the equation for $f \in L^p(\omega^n)$, $p > n/m$, and the bound on p is sharp.

Geometric applications

- Fu-Yau '08 proved the existence of some special solutions to the Hull-Strominger system (supersymmetry in string theory) reducing the system to M-A type equation in dimension 2. Those solutions produce a toric fibration over a $K3$ surface. In higher dimensions such a reduction leads to hessian type equation with a gradient term:

$$(\chi(z,u) + dd^c u)^k \wedge \omega^{n-k} = \psi(z,u,\nabla u)\omega^n,$$

where $\chi(z,u)$ is a (1,1) form. Phong-Picard-Zhang '14-'19 solved those equations under mild assumptions.

- For a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ the graph $(x, \nabla u(x))$ is called special Lagrangian if it is area minimizing among all submanifolds with the same boundary. Harvey and Lawson proved that the gradient graph is area minimizing iff the eigenvalues of the hessian of u satisfy the equation

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- In relation to the mirror symmetry for Calabi-Yau manifolds Collins-Jacob-Yau '18 solved a corresponding equation assuming the existence of subsolutions. If ω is a Kähler form, we look for a real smooth, closed $(1,1)$ form χ such that

$$\Im((\omega + i\chi)^n) = const.\Re((\omega + i\chi)^n).$$

The solution yields special Lagrangian surfaces in Calabi-Yau manifolds.

Hessian equations on Hermitian manifolds

- Now ω is not closed on a compact complex mfd X. We wish to find a constant c and (m,ω)-sh function u such that

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = cf\omega^n.$$

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- Existence in the smooth non-degenerate case is due to Tosatti - Weinkove for the M-A equation. In general to D. Zhang and G. Székelyhidi independently.

Previous results

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- **Main theorem.** *Let* $0 \leq f \in L^p(X, \omega^n), p > n/m$, *and* $\int_X f\omega^n > 0$. *There exist a continuous* (ω, m)*-subharmonic function* u *and a constant* $c > 0$ *satisfying*

$$H_m(u) = (\omega + dd^c u)^m \wedge \omega^{n-m} = cf\omega^n.$$

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$$H_m(u) = (\omega + dd^c u)^m \wedge \omega^{n-m} = cf\omega^n.$$

- We also obtained stability estimates and showed the existence of decreasing sequences of smooth approximants of a continuous (ω, m)-sh function in this class. This is nontrivial in this setting and requires "Berman trick". However we were not able to deal with **bounded** (ω, m)-sh functions.

Results

- First we show that the complex Hessian measure of a bounded $m-\omega$-sh function u is well defined as the weak limit of

$$H_m(u) := \lim_{\delta\to 0} H_m(u^\delta) = \lim_{\delta\to 0}(dd^c u^\delta)^m \wedge \omega^{n-m},$$

  where $\{u^\delta\}$ is a sequence of smooth $m-\omega$-sh functions converging decreasingly to u (limit is independent of the approximating sequence).
  The proof is based on the CLN inequality in the previous paper and fairly long argument in several steps.

Lemma

Let $u_1, ..., u_p$ with $1 \leq p \leq m-1$ be bounded $m-\omega$-sh functions. Assume $T_0 = 1$ and $T_{p-1} = dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1$. Then, the current

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 := dd^c[u_p T_{p-1}]$$

is a well-defined current of order zero whose coefficients are signed Radon measures. Moreover,

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \omega^{n-m}$$

is a positive $(n-m+p, n-m+p)$-current.

- This is the starting point for the desired definition. For psh functions the currents above are positive, in our case we only have currents of order zero. The currents are obtained via approximating sequences. The limits should be independent of the approximants we choose.

Results

- To verify this we need several statements.

Theorem (CLN inequality)

Let $K \subset\subset U \subset\subset \Omega$, where K is compact and U is open. Let $u_1, \ldots, u_p$, be bounded $m-\omega$-sh functions in Ω, where $1 \leq p \leq m$. Then, there exists a constant C depending on K, U, ω such that
$\int_K dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-p} \leq C \left(1 + \sum_{s=1}^{p} \|u_s\|_{L^\infty(U)}\right)^p$.

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- The following lemma is a key to get the convergence results for m-sh functions.

Results

- "Integration by parts inequalities":

Lemma

Let $\{v_l\}_{l\geq 1}$ be a sequence of smooth functions from P^* (m-sh functions which are psh close do the boundary) and let $w \in P^*$ be smooth. Assume $v_l \leq w$ and $v_l \downarrow v \in P^*$. Let ρ be a bounded $m-\omega$-sh in Ω such that $-1 \leq \rho \leq 0$. Then, for $\eta = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-2}$ with $u_1, ..., u_{m-2} \in P^*$

$$\int_\Omega (w-v)dd^c\rho\wedge\eta\wedge\omega^{n-m+1} \leq C\left(\int_\Omega (w-v)dd^c v\wedge\eta\wedge\omega^{n-m+1}\right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\right)^{\frac{1}{2}} + C\left(\int_\Omega (w-v)\eta\wedge\omega^{n-m+2}\right)^{\frac{1}{4}}.$$

with C depending on P^* and sup norms of involved functions.

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$$\int_\Omega (w-v) dd^c \rho \wedge \eta \wedge \omega^{n-m+1} \leq C \left(\int_\Omega (w-v) dd^c v \wedge \eta \wedge \omega^{n-m+1} \right)^{\frac{1}{2}}$$

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Results

- Then the definition of $dd^c u_1 \wedge \cdots \wedge dd^c u_p$ is finally proven to be correct and so does the symmentry of this product w.r.t. bounded $m - \omega$-sh functions. This allows to mimic pluripotential theory, in particular to define the Bedford-Taylor capacity:

$$cap_m(E) = \sup\left\{\int_E H_m(u) : u \text{ is } m - \omega\text{-sh in } \Omega, -1 \leq u \leq 0\right\}.$$

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- We prove quasi-continuity of $m-\omega$-sh functions with respect to this capacity.
- The capacity IS NOT realized by the relative extremal function. However we have a characterization of a polar set E by $cap_m^*(E) = 0$.
- Using this we obtain the equivalence of polar sets and negligible sets (famous Lelong's problem for psh functions).

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Lemma

There exists a uniform constant C, depending on n, m, such that the following inequalities are satisfied.

(A) *For $1 \leq i \leq m-1$ and $\lambda \in \Gamma_m$*

$$\frac{\lambda_1 \cdots \lambda_m}{\lambda_i} \leq C(S_{m-1;i}\, S_{m-1})^{\frac{1}{2}}.$$

(B) *Generally, for $1 \leq \ell \leq n$ and increasing multi-indices $(i_1, ..., i_{m-1})$,*

$$\prod_{i_s \neq \ell; s=1}^{m-1} |\lambda_{i_s}| \leq C\left(S_{m-1;\ell} S_{m-1}\right)^{\frac{1}{2}}.$$

Results

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Lemma

There exists a uniform constant C depending on ω, n, m such for every $\gamma \in \Gamma_m(\omega)$ $(\gamma^k \wedge \omega^{n-k}(z) > 0 \quad \text{for } k = 1, ..., m)$

$$\left| \int \phi\psi \, dh \wedge d^c\omega \wedge \gamma^{m-1} \wedge \omega^{n-m-1} \right|^2$$
$$\leq C \int |\phi|^2 \, dh \wedge d^c h \wedge \gamma^{m-1} \wedge \omega^{n-m} \times \int |\psi|^2 \, \gamma^{m-1} \wedge \omega^{n-m+1}.$$

- (here $\gamma^{m-1} \wedge \omega^{n-m-1}$ is not necessarily positive, h, ϕ, ψ are smooth functions).

Results

- A version of Cauchy-Schwarz:

Lemma

There exists a uniform constant C depending on ω, n, m such for every $\gamma \in \Gamma_m(\omega)$ $(\gamma^k \wedge \omega^{n-k}(z) > 0 \quad$ for $k = 1, ..., m)$

$$\left| \int \phi\psi \, dh \wedge d^c\omega \wedge \gamma^{m-1} \wedge \omega^{n-m-1} \right|^2$$

$$\leq C \int |\phi|^2 \, dh \wedge d^c h \wedge \gamma^{m-1} \wedge \omega^{n-m} \times \int |\psi|^2 \, \gamma^{m-1} \wedge \omega^{n-m+1}.$$

- (here $\gamma^{m-1} \wedge \omega^{n-m-1}$ is not necessarily positive, h, ϕ, ψ are smooth functions).
- More complicated integral estimates.

THANK YOU