Topological invariants and Holomorphic mappings

Kang-Tae Kim

Department of Mathematics
POSTECH, South Korea

Monday, 21 October 2024
Wuppertal, Germany

R. E. Greene, K.-T. Kim and N. V. Shcherbina:
“Topological invariants and Holomorphic Mappings”

Comptes Rendus Mathématique (Institut de France),
Vol. 360 (2022) p. 829-844.

https://doi.org/10.5802/crmath.336

ℓ_1-invariant

$\Omega \subset \mathbb{C}^n$, a Kobayashi hyperbolic domain. γ a piecewise C^1 curve in Ω. $L(\gamma) =$ the Kobayashi length of γ. Then

$$\ell_1(\Omega) := \inf\{L(\alpha) \colon \alpha \not\sim *\}.$$

ℓ_1 for annuli

$$\mathrm{Ann}\,(0, R) = \left\{ z \in \mathbb{C} \colon \frac{1}{\sqrt{R}} < |z| < \sqrt{R} \right\}$$

$$\ell_1 \left(\mathrm{Ann}\,(0, R) \right) = \frac{\pi^2}{\ln R},$$

(Note: the conformal modulus.)

Theorem (Hadamard 1890±)

The Annuli $A(0, r, R), A(0, s, S)$ *in the complex plane are conformally equivalent if, and only if,* $r/R = s/S$.

J. Hadamard; R. de Possel

Theorem (Hadamard 1890$\pm$)

The Annuli $A(0, r, R), A(0, s, S)$ in the complex plane are conformally equivalent if, and only if, $r/R = s/S$.

Theorem (R. de Possel, 1933)

For annuli $A(0, r, R), A(0, s, S)$, there exists an injective holomorphic map $f: A(0, r, R) \to A(0, s, S)$ such that $f(A(0, r, R))$ separates the boundaries of $A(0, s, S)$, if and only if $R/r \leq S/s$.

An extension

Theorem (Gr.-K.-Shch.)

For hyperbolic manifolds M and N with $\pi_1(M) \neq 0$, if there is a holomorphic map $f: M \to N$ induces $f_\star : \pi_1(M) \to \pi_1(N)$, an injection on homotopy, then $\ell_1(M) \geq \ell_1(N)$.

Tubular neighborhoods

For $r > 0$ sufficiently small,

$$T^n(r) := \bigcup_{t\in\mathbb{R}} \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1 - e^{it}|^2 + |z_2|^2 + \ldots + |z_n|^2 < r^2\}.$$

Theorem

If $f\colon T^n(r) \to T^m(s)$ is holomorphic and $0 < s < r < 1$, then f is homotopic to a constant map.

ℓ_k-invariant, $k > 1$

- (X, d) a metric space. $\delta(A) = \sup_{p,q \in A} d(p, q)$.
  Recall the Hausdorff k-measure for (X, d) is defined by

$$\mu_d^k(A) = \sup_{\epsilon > 0} \inf\{\sum_{i=1}^{\infty} \left(\delta(A_i)\right)^k : A = \bigcup A_i, \delta(A_i) < \epsilon\}.$$

- Call a continuous map $f: S^k \to U$ **Hausdorff-Kobayashi** k**-rectifiable** if

$$\mu^k_{\mathrm{Kob},U}\left(f(S^k)\right) < \infty.$$

- We define:

$$\ell_k(U) := \text{the inf of H-K } k\text{-measures of all the}$$
$$\text{H-K rectifiable representatives of the nontrivial}$$
$$\text{free homotopy classes of } f: S^k \to U.$$

Tubular neighborhoods of S^k, $k > 1$

Lemma

S^k, C^2, totally real in $\mathbb{C}^n$ and contained in a bounded domain $\Omega \subset \mathbb{C}^n$. Assume that $S^k \not\sim *$. There exists R such that for any $r \in (0, R)$,

$$T_r := \bigcup_{x \in S^k} B^n(x, r)$$

satisfies $\ell_k(T_r) > 0$.

Moreover, $\cdots$

For s, r_2 sufficiently small, the following hold:

Theorem

Let $U \subset \mathbb{C}^m$, $V \subset \mathbb{C}^m$ be bounded domains with $\pi_1(U) \neq 0$. If there is a holomorphic $f : U \to V$ injective on homotopy, then $\ell_k(T_r) \geq \ell_k(T_s)$.

For s, r_2 sufficiently small, the following hold:

Theorem

Let $U \subset \mathbb{C}^m$, $V \subset \mathbb{C}^m$ be bounded domains with $\pi_1(U) \neq 0$. If there is a holomorphic $f \colon U \to V$ injective on homotopy, then $\ell_k(T_r) \geq \ell_k(T_s)$.

Corollary

If $r_1 > r_2$ and if $f \colon T_{r_1} \to T_{r_2}$ is holomorphic, then f is homotopic to a constant map.

Tubular neighborhoods of general M; Degrees

Let M a compact connected smooth (C^2 suffices) submanifold without boundary of $\mathbb{R}^N$.

($r > 0$ sufficiently small). $g : M \to T_r(M)$ smooth.

$\pi : T_r(M) \to M$ the orthogonal projection. Then define by

$$\deg(g) := \deg(\pi \circ g)$$

Tubular neighborhoods of general M; Degrees

Let M a compact connected smooth (C^2 suffices) submanifold without boundary of $\mathbb{R}^N$.

($r > 0$ sufficiently small). $g : M \to T_r(M)$ smooth.

$\pi\colon T_r(M) \to M$ the orthogonal projection. Then define by

$$\deg(g) := \deg(\pi \circ g)$$

If $G\colon T_{r_1} \to T_{r_2}$ continuous, then define by

$$\deg G := \deg(G \circ incl.).$$

(If M is not orientable, use $\mathbb{Z}_2$-degree.)

On Degrees

Theorem

Let M a C^2, compact, connected, totally real submanifold of $\mathbb{C}^n$. Then there exists $R > 0$ such that, if $0 < r < s < R$ and if $F\colon T_s(M) \to T_r(M)$ is holomorphic then, $\deg F = 0$.

Contractability of bounded domains

Theorem

$U \subset \mathbb{C}^n$, a bounded domain with C^2 boundary. If $\exists f: U \to U$ holomorphic with $f(U) \Subset U$ with $f_* : \pi_k(U, z) \to \pi_k(U, f(z))$ isomorphic, for all $k = 1, 2, \ldots$, then U is contractible.

A technique

Lemma (GKShch, Sect. 9)

If U, V bounded domains in $\mathbb{C}^n$ with $U \Subset V$ then there exists $0 < c < 1$ such that

$$F_V^{Kob}(p, v) \leq c\, F_U^{Kob}(p, v), \ \forall (p, v) \in U \times \mathbb{C}^n.$$