Invariant metrics and rescaling of automorphism sequences

Jae-Cheon Joo

Department of Mathematics, KFUPM, KSA

October 22, 2024

Frankel's theorem

Theorem (S. Frankel, 1989)

A bounded (Kobayashi hyperbolic) convex domain in $\mathbb{C}^n$ which covers a compact complex manifold is a bounded symmetric domain.

- Analytic Part about normalizing the domain by a "scaling" sequence and obtaining a 1-parameter subgroup of automorphisms.
- Algebraic Part to prove a bounded domain which covers a compact manifold with 1-parameter subgroup of automorphisms has a semisimple automorphism group.

Rescaling sequence

Let Ω be a domain in $\mathbb{C}^n$ and $p \in \Omega$ fixed. Let $\{\varphi_j\}$ be a sequence of automorphisms. Let

$$\psi_j(z) := [d\varphi_j(p)]^{-1}(\varphi_j(z) - \varphi_j(p)). \quad (1)$$

Frankel proved that if Ω is hyperbolic and convex, then

- $\psi_j \to \hat{\psi}$ and $\hat{\psi} : \Omega \to \hat{\psi}(\Omega)$ is a biholomorphism.
- If moreover, Ω covers a compact manifold, $\exists\{\varphi_j\}$ such that $\varphi_j(p)$ tends to a "regular" boundary point.
- $\hat{\psi}(\Omega)$ (and hence Ω too) admits a 1-parameter subgroup of automorphisms.

Questionss

Question 1. Can we replace the stretching factor of (1) with one which depends only on the geometry of Ω and $p_j = \varphi_j(p)$?

- Useful for investigating asymptotic behavior of invariants.
- For bounded convex domains, the stretching factor can be replaced by that of Pinchuk's sequence (Kim-Krantz).
- If Ω is a bounded domain in $\mathbb{C}^2$ with smooth boundary of finite type, then both Pinchuk's and Frankel's sequences can be modified by the composition with entire automorphisms to be convergent (S. Joo).

Question 2. Can we describe a geometric condition which guarantee the convergence of (1)?

Finsler metrics

Let Ω be a domain in $\mathbb{C}^n$. A (complex Finsler) metric on Ω is a usc function $F : \Omega \times \mathbb{C}^n \to \mathbb{R}$ such that

- $F(z, v) \geq 0$ for all $z \in \Omega$ and $v \in \mathbb{C}^n$ and $F(z, v) = 0$ only if $v = 0$.
- $F(z, \lambda v) = |\lambda| F(z, v)$ for any $\lambda \in \mathbb{C}$.

For a metric F on Ω,

$$I_F(z) := \{v \in \mathbb{C}^n : F(z, v) < 1\}$$

is called the indicatrix of F at z.

A metric F is said to be

- convex if $I_F(z)$ is a convex set for all $z \in \Omega$.
- invariant if
$$F(z, v) = F(\varphi(z), d\varphi(z)(v))$$
for any $z \in \Omega$, $v \in \mathbb{C}^n$ and automorphism φ.

Construction of Stretching Factor

For $z \in \Omega$, let $\mu_1 = \max\{F(z,v) : |v| = 1\}$ and u_1 a unit vector such that $F(z,u_1) = \mu_1$. Choose μ_2 and $u_2 \perp u_1$ by

$$\mu_2 = \max\{F(z,v) : |v| = 1,\ v \perp u_1\}, \quad F(z,u_2) = \mu_2.$$

Continuing this process, we can find a unitary basis $u_1, \ldots, u_n$ of $\mathbb{C}^n$ and $\mu_1, \ldots, \mu_n$.

Let $L = L_z$ be the lineat map on $\mathbb{C}^n$ defined by

$$L(u_\alpha) = \mu_\alpha e_\alpha, \quad \alpha = 1, \ldots, n$$

Theorem (H. Alzaki - J.)

Let $L_j = L_{\varphi_j(p)}$. Then (1) is convergent if and only if

$$\sigma_j := L_j(\varphi_j(\cdot) - \varphi_j(p)),$$

where F is an invariant convex metric, is convergent.

- It suffices to show there exists $C = C(\Omega, p) > 0$ such that

  $$C^{-1}|[d\varphi_j(p)]^{-1}(v)| \leq |L_j v| \leq C|[d\varphi_j(p)]^{-1}(v)|, \text{ for all } j \text{ and } v.$$

- The lower estimate is almost obvious from the construction of L_j.
- For the upper bound, by Cramer's rule, we need to estimate the volume of indicatrix at each p_j. This is why the convexity of metric is needed.

Convergence of Frankel's Sequence

Let $d_{\Omega,F}$ be the induced distance function on Ω by F. For $r > 0$, we denote the $d_{\Omega,F}$-ball of radius r centered at z by $B_{\Omega,F}(z;r)$, that is,

$$B_{\Omega,F}(z;r) = \{w \in \Omega : d_{\Omega,F}(z,w) < r\}.$$

We say that Ω is uniformly dominated by F if

- $d_{\Omega,F}$ is complete.
- there is a positive function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$B_{\Omega,F}(z;r) - z \subset \lambda(r) I_{\Omega,F}(z)$$

for any $z \in \Omega$.

Convergence of Frankel's Sequence

Theorem (H. Alzaki - J.)

(a) *If there is an invariant mtric F which uniformly dominates Ω, then (1) is convergent.*

(b) *Any Kobayashi hyperbolic convex domain is uniformly dominated by the Kobayashi-Royden metric*

Proof/Sketch.

- (a) is an immediate consequence of the Montel theorem.
- For (b), observe at first that the upper half plane H in $\mathbb{C}$ is uniformly dominated by the Poincaré metric. In this case, $\lambda(r)$ can be chosen by

$$\lambda(r) = \frac{\tanh(r/2)}{1 - \tanh(r/2)} =: \lambda_H(r).$$

Convergence of Frankel's Sequence

- Notice that the uniform domination property is not holomorphic invariant but still affine invariant.
- For any $z \in \Omega$ and nonzero $v \in \mathbb{C}^n$, let $L_{z,v}$ be the affine complex line through z in direction of v. Let
  - $d = \min\{|z - w| : w \in \partial(\Omega \cap L_{z,v})\}$
  - $p \in \partial(\Omega \cap L_{z,v})$ be a point of minimum.
- We can cunstruct an affine map A such that
  - $A(\Omega) \subset H \times \mathbb{C}^{n-1}$,
  - $A(\Omega \cap L_{z,v}) \subset H \times \{0\}$,
  - $A(z) = \sqrt{-1}d$,
  - $A(p) = 0$.

Convergence of Frankel's Sequence

- For any $w \in \Omega \cap L_{z,v}$,
  - $d_\Omega(z,w) \geq d_{H\times\mathbb{C}^{n-1}}(z,w) = d_H(z,w)$,
  - $K_\Omega(z;v) \leq \frac{|v|}{d} = 2K_H(\sqrt{-1}d;v)$.
- Therefore,

$$(B_\Omega(z,r) - z) \cap L_{z,v} \subset 2\lambda_H(r)(I_\Omega(z) \cap L_{z,v}).$$

- The conclusion follows since z and v are arbitrary.

Maximal Circularity

If Ω is a BSD, then $\partial\Omega$ is piecewise smooth and each regular boundary point is included in an analytic variety which lies on $\partial\Omega$ unless Ω is the ball.

A boundary point is called minimal if there is no nontrivial analytic variety through the point in $\partial\Omega$.

Usually, the linit of scaling is more canoncal when $\varphi_j(p)$ tends to a minimal boundary point.

Question. What kind of symmetry can be expected if $\varphi_j(p)$ tends to a minimal boundary point?

A domain Ω is said to be

- circular if $e^{i\theta}z \in \Omega$ for any $\theta \in \mathbb{R}$ whenever $z \in \Omega$.
- completely circular or balanced if $cz \in \Omega$ for any $c \in \Delta$ whenever $z \in \overline{\Omega}$.

Remark. Any BSD is completely circular. Indeed, the circularity is a signature property of BSD among homogeneous domains.

Question 3. If $\varphi_j(p)$ tends to a minimal boundary point of convex Ω, then Ω is biholomorphic to a complete circular domain??

Maximal Circularity

In order to study this problem geometrically, we define a function $c_\Omega : \Omega \to \mathbb{R}$ on a bounded domain Ω which we call the maximal circularity by

$$c_\Omega(z) = \sup\{r/R : \exists \varphi \in \mathcal{B}_\Omega(z),\ rI_{\varphi(\Omega)}(0) \subset \varphi(\Omega) \subset RI_{\varphi(\Omega)}(0)\}$$

where $\mathcal{B}_\Omega(z)$ is the set of all bounded holomorphic embedding φ into $\mathbb{C}^n$ with $\varphi(z) = 0$.

- $0 < c_\Omega(z) \leq 1$.
- $c_\Omega(0) = 1$ if Ω is a bounded pseudoconvex balanced domain (T. J. Barth).
- Conversely, if $c_\Omega(z) = 1$ for some $z \in \Omega$, Ω is biholomorphic to $I_\Omega(z)$.

Question 4. Let Ω be bounded and convex. $q \in \partial\Omega$ is minimal. Let p_j be a sequence in Ω which tends to q. Then $c_\Omega(p_j) \to 1$ as $j \to \infty$?

Theorem (H. Alzaki - J.)

(a) *If Ω is bounded strongly pseudoconvex domain, then $c_\Omega(p) \to 1$ as $p \to \partial\Omega$.*

(b) *If Ω is a bounded convex normal analytic polyhedral domain, then $c_\Omega(p) \to 1$ as p tends to a minimal boundary point.*

Let D be any bounded balanced domain in $\mathbb{C}^n$. For a domain $\Omega \subset \mathbb{C}^n$, the D-squeezing function is defined by

$$s_\Omega^D(z) := \sup\{r > 0 : \exists \varphi \in \mathcal{B}_\Omega(z),\ rD \subset \varphi(\Omega) \subset D\}$$

for $z \in \Omega$. Then

- For any bounded pseudoconvex balanced domain D,

$$c_\Omega(z) \geq \left(s_\Omega^D(z)\right)^2.$$

- Therefore, (a) is a consequence of the boundary behavior of $s_\Omega^{\mathbb{B}^n}$ (Kim-Zhang, Diederich-Fornaess-Wold).
- (b) can be obtained by the asymptotic behavior of $s_\Omega^{\Delta^n}$.

Further Question

Let p be a minimal boundary point of a bounded convex domain Ω. Let $\{p_j\}$ be a sequence in Ω approaching p. Is there a bounded balanced domain $\widehat{D}$ such that

$$\limsup_{j\to\infty} s_{\Omega}^{\widehat{D}}(p_j) \geq \limsup_{j\to\infty} s_{\Omega}^{D}(p_j)$$

for any other balanced domain D?

Guess. The Hausdorff limit of $L_j(I_\omega(p_j))$ may work. Or if we denote by M_z the maximal balanced domain such that $z + M_z$ is contained in Ω (called the maximal indicatrix), then it turns out M_z is convex for any $z \in \Omega$ if and only if Ω is convex (Nikolov-Thomas). The Hausdorff limit of $L_j(M_{p_j})$ is so well-defined and it can be also a candidate.

Thank You!