Invariant subspaces for finite index shifts and the invariant subspace problem in Hilbert spaces

Filippo Bracci

Dipartimento di Matematica
Università di Roma “Tor Vergata”

Wuppertal, 21-25 October 2024

The invariant subspace problem

Invariant subspace problem: Let H be a Hilbert space of dimension greater than 1. Let $T : H \to H$ be a linear and bounded operator. Does there exist a *closed* subspace $\{0\} \subsetneq V \subsetneq H$ such that $T(V) \subseteq V$?

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

- H finite dimensional.

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

- H finite dimensional.
- H not separable.

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

- H finite dimensional.
- H not separable.
- T normal operator.

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

- H finite dimensional.
- H not separable.
- T normal operator.
- T (polynomially) compact operator.

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

- H finite dimensional.
- H not separable.
- T normal operator.
- T (polynomially) compact operator.
- T Lomonosov (i.e. commuting with a non-trivial compact operator).

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

- H finite dimensional.
- H not separable.
- T normal operator.
- T (polynomially) compact operator.
- T Lomonosov (i.e. commuting with a non-trivial compact operator).
- Several other particular operators.

The invariant subspace problem

The answer is known to be YES in case (Von Neumann, Halmos, Bernstein and Robinson, ...):

- H finite dimensional.
- H not separable.
- T normal operator.
- T (polynomially) compact operator.
- T Lomonosov (i.e. commuting with a non-trivial compact operator).
- Several other particular operators.

In Banach (not Hilbert) spaces there are examples with negative answer (Enflo, Reid).

Defect of an operator

Definition

Let $T : H \to H$ be a linear bounded operator. The *defect* of T is

$$\delta(T) := \dim \overline{(I - T^*T)H}$$

Defect of an operator

Definition

Let $T : H \to H$ be a linear bounded operator. The *defect* of T is

$$\delta(T) := \dim \overline{(I - T^* T)H}$$

$\delta(T) = 0$ if and only if T isometry.

Invariant subspace problem and defect

Theorem (Corollary to Beurling theorem)

The invariant subspace problem has affirmative answer for operators T with $\delta(T) \leq 1$.

Invariant subspace problem and defect

Theorem (Corollary to Beurling theorem)

The invariant subspace problem has affirmative answer for operators T with $\delta(T) \leq 1$.

Theorem (Br.-Gallardo Gutierrez, 2024)

The invariant subspace problem has affirmative answer for operators T with $\delta(T) < +\infty$.

Invariant subspace problem and defect

Theorem (Corollary to Beurling theorem)

The invariant subspace problem has affirmative answer for operators T with $\delta(T) \leq 1$.

Theorem (Br.-Gallardo Gutierrez, 2024)

The invariant subspace problem has affirmative answer for operators T with $\delta(T) < +\infty$.

If one can extend to $\delta(T) = +\infty$ the invariant subspace problem is solved.

The shift

Definition

A linear operator $S : H \to H$ is a *shift* if S is an isometry and $\|(S^*)^n v\| \to 0$ for all $v \in H$.The *index* $\iota(S)$ of a shift is the dimension of the kernel of S^*.

The shift

Definition

A linear operator $S : H \to H$ is a *shift* if S is an isometry and $\|(S^*)^n v\| \to 0$ for all $v \in H$.The *index* $\iota(S)$ of a shift is the dimension of the kernel of S^*.

$T : H \to H$ and $R : \tilde{H} \to \tilde{H}$ are *isometrically equivalent* if there exists an isometric isomorphism $U : H \to \tilde{H}$ such that $T = U^{-1}RU$.

The shift

Definition

A linear operator $S : H \to H$ is a *shift* if S is an isometry and $\|(S^*)^n v\| \to 0$ for all $v \in H$.The *index* $\iota(S)$ of a shift is the dimension of the kernel of S^*.

$T : H \to H$ and $R : \tilde{H} \to \tilde{H}$ are *isometrically equivalent* if there exists an isometric isomorphism $U : H \to \tilde{H}$ such that $T = U^{-1}RU$.

Proposition

Every two shifts of the same index are isometrically equivalent.

The shift

Theorem (Universal model)

Let $T : H \to H$ be a bounded and linear operator. Let $S : \tilde{H} \to \tilde{H}$ be a shift. If $\delta(T) \leq \iota(S)$ then there exists a closed S^*-invariant subspace V such that T is isometrically equivalent to $S^*|_V$.

Let S be any shift.

Let S be any shift. Then

S^* has no *minimal* closed invariant subspaces of dimension > 1 if and only if the answer to the invariant subspace problem is affirmative for operators with defect $\leq \iota(S)$.

Let S be any shift. Then

S^* has no *minimal* closed invariant subspaces of dimension > 1 if and only if the answer to the invariant subspace problem is affirmative for operators with defect $\leq \iota(S)$.

or equivalently

S has no *maximal* closed invariant subspaces of co-dimension > 1 if and only if the answer to the invariant subspace problem is affirmative for operators with defect $\leq \iota(S)$.

The shift on the Hardy space $H^2(\mathbb{D})$

Let

$$H^2(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}, \mathbb{C}) : \|f\|^2_{H^2(\mathbb{D})} := \limsup_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < +\infty\}$$

The shift on the Hardy space $H^2(\mathbb{D})$

Let

$$H^2(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}, \mathbb{C}) : \|f\|^2_{H^2(\mathbb{D})} := \limsup_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < +\infty\}$$

The shift $S_1 : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is given by

$$S_1(f(z)) := zf(z).$$

$\iota(S_1) = 1.$

Beurling Theorem

A holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ is *inner* if $|\phi^*(e^{i\theta})| = 1$ for a.e. $\theta \in [0, 2\pi]$.

Beurling Theorem

A holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ is *inner* if $|\phi^*(e^{i\theta})| = 1$ for a.e. $\theta \in [0, 2\pi]$.

Theorem (Beurling)

A closed subspace $\{0\} \subsetneq M \subseteq H^2(\mathbb{D})$ is S_1-invariant if and only if there exists a inner function ϕ such that

$$M = \phi H^2(\mathbb{D}) := \{\phi \cdot f : f \in H^2(\mathbb{D})\}.$$

Beurling Theorem

A closed S_1-invariant subspace $\phi H^2(\mathbb{D}) = H^2(\mathbb{D})$ if and only if ϕ is constant.

Beurling Theorem

A closed S_1-invariant subspace $\phi H^2(\mathbb{D}) = H^2(\mathbb{D})$ if and only if ϕ is constant.

A closed S_1-invariant subspace $\phi H^2(\mathbb{D})$ has codimension 1 if and only if ϕ is an automorphism of $\mathbb{D}$.

Beurling Theorem

A closed S_1-invariant subspace $\phi H^2(\mathbb{D}) = H^2(\mathbb{D})$ if and only if ϕ is constant.

A closed S_1-invariant subspace $\phi H^2(\mathbb{D})$ has codimension 1 if and only if ϕ is an automorphism of $\mathbb{D}$.

If ϕ is not constant nor an automorphism, there exist two nonconstant inner functions ϕ_1, ϕ_2 such that $\phi = \phi_1\phi_2$.

Beurling Theorem

A closed S_1-invariant subspace $\phi H^2(\mathbb{D}) = H^2(\mathbb{D})$ if and only if ϕ is constant.

A closed S_1-invariant subspace $\phi H^2(\mathbb{D})$ has codimension 1 if and only if ϕ is an automorphism of $\mathbb{D}$.

If ϕ is not constant nor an automorphism, there exist two nonconstant inner functions ϕ_1, ϕ_2 such that $\phi = \phi_1\phi_2$. Thus, $\phi H^2(\mathbb{D}) \subsetneq \phi_1 H^2(\mathbb{D})$.

Beurling Theorem

A closed S_1-invariant subspace $\phi H^2(\mathbb{D}) = H^2(\mathbb{D})$ if and only if ϕ is constant.

A closed S_1-invariant subspace $\phi H^2(\mathbb{D})$ has codimension 1 if and only if ϕ is an automorphism of $\mathbb{D}$.

If ϕ is not constant nor an automorphism, there exist two nonconstant inner functions ϕ_1, ϕ_2 such that $\phi = \phi_1\phi_2$. Thus, $\phi H^2(\mathbb{D}) \subsetneq \phi_1 H^2(\mathbb{D})$.

Corollary

S_1 does not have maximal closed invariant subspaces of codimension > 1.

Beurling Theorem

A closed S_1-invariant subspace $\phi H^2(\mathbb{D}) = H^2(\mathbb{D})$ if and only if ϕ is constant.

A closed S_1-invariant subspace $\phi H^2(\mathbb{D})$ has codimension 1 if and only if ϕ is an automorphism of $\mathbb{D}$.

If ϕ is not constant nor an automorphism, there exist two nonconstant inner functions ϕ_1, ϕ_2 such that $\phi = \phi_1 \phi_2$. Thus, $\phi H^2(\mathbb{D}) \subsetneq \phi_1 H^2(\mathbb{D})$.

Corollary

S_1 does not have maximal closed invariant subspaces of codimension > 1. In particular the invariant subspace problem has affirmative answer for operators with defect ≤ 1.

The Beurling-Lax Theorem

Beurling's theorem can be reinterpreted in terms of operators:

Theorem

A subspace $\{0\} \subsetneq M \subseteq H^2(\mathbb{D})$ is a closed S_1-invariant subspace if and only if there exists an isometry $U : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ such that $M = U(H^2(\mathbb{D}))$.

The Beurling-Lax Theorem

Beurling's theorem can be reinterpreted in terms of operators:

Theorem

A subspace $\{0\} \subsetneq M \subseteq H^2(\mathbb{D})$ is a closed S_1-invariant subspace if and only if there exists an isometry $U : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ such that $M = U(H^2(\mathbb{D}))$.

Indeed, the only isometries of $H^2(\mathbb{D})$ are given by multiplication by inner functions.

The Beurling-Lax Theorem

Beurling's theorem can be reinterpreted in terms of operators:

Theorem

A subspace $\{0\} \subsetneq M \subseteq H^2(\mathbb{D})$ is a closed S_1-invariant subspace if and only if there exists an isometry $U : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ such that $M = U(H^2(\mathbb{D}))$.

Indeed, the only isometries of $H^2(\mathbb{D})$ are given by multiplication by inner functions.
Note that the multiplication operator commutes with S_1.

The Beurling-Lax Theorem

Theorem (Beurling-Lax)

Let $S : H \to H$ be a shift. A subspace $\{0\} \subsetneq M \subseteq H$ is a closed S-invariant subspace if and only if there exists a bounded linear operator $A : H \to H$ (called a S-inner operator) such that

1. $A \circ S = S \circ A$,
2. *If* $H = \ker A \overset{\perp}{\oplus} V$ *then* $A|_V$ *is an isometry.*
3. $M = A(H)$.

Indeed, the only isometries of $H^2(\mathbb{D})$ are given by multiplication by inner functions.

The Hilbert space $H^2(\mathbb{D}) \oplus \ldots \oplus H^2(\mathbb{D})$

Let

$$\mathbb{H} := H^2(\mathbb{D})e_1 \oplus \ldots \oplus H^2(\mathbb{D})e_d,$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the j-th position, $j = 1, \ldots, d$.

The Hilbert space $H^2(\mathbb{D}) \oplus \ldots \oplus H^2(\mathbb{D})$

Let
$$\mathbb{H} := H^2(\mathbb{D})e_1 \oplus \ldots \oplus H^2(\mathbb{D})e_d,$$
where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the j-th position, $j = 1, \ldots, d$.

The shift $S : \mathbb{H} \to \mathbb{H}$ is given by
$$S(f_1e_1 + \ldots + f_de_d) = S_1(f)e_1 + \ldots + S_1(f_d)e_d.$$

$\iota(S) = d$.

Determinantal operators

Let $1 \leq m \leq d$ be an integer number and let $A = (a_{jk})$ be a $m \times m$ matrix whose entries are bounded holomorphic functions in $\mathbb{D}$ (that is, elements of $H^\infty(\mathbb{D})$). Let $1 \leq s_1 < \ldots < s_m \leq d$. Let $j \in \{1, \ldots, m\}$. A *determinantal operator* is any linear operator $L : \mathbb{H} \to H^2(\mathbb{D})$ of the form

Determinantal operators

Let $1 \leq m \leq d$ be an integer number and let $A = (a_{jk})$ be a $m \times m$ matrix whose entries are bounded holomorphic functions in $\mathbb{D}$ (that is, elements of $H^\infty(\mathbb{D})$). Let $1 \leq s_1 < \ldots < s_m \leq d$. Let $j \in \{1, \ldots, m\}$. A *determinantal operator* is any linear operator $L : \mathbb{H} \to H^2(\mathbb{D})$ of the form

$$L(f_1 e_1 + \ldots + f_d e_d) := \det \begin{pmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{j-1,1} & \ldots & a_{j-1,m} \\ f_{s_1} & \ldots & f_{s_m} \\ a_{j+1,1} & \ldots & a_{j+1,m} \\ \vdots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mm} \end{pmatrix}.$$

Determinantal operators

Any determinantal operator $L : \mathbb{H} \to H^2(\mathbb{D})$ is bounded and commutes with S.

Determinantal operators

Any determinantal operator $L : \mathbb{H} \to H^2(\mathbb{D})$ is bounded and commutes with S.

For $d = 1$, i.e., $\mathbb{H} = H^2(\mathbb{D})$, there is only one determinantal operator, $L = \mathrm{id}$.

Determinantal bricks

Let φ be either identically zero or an inner function.

A *determinantal brick* Q_φ *based on* φ is any set of the form

$$Q_\varphi := L^{-1}(\varphi H^2(\mathbb{D})) = \{F \in \mathbb{H} : L(F) \in \varphi H^2(\mathbb{D})\},$$

where L is a determinantal operator.

Determinantal bricks

Let φ be either identically zero or an inner function.

A *determinantal brick* Q_φ *based on* φ is any set of the form

$$Q_\varphi := L^{-1}(\varphi H^2(\mathbb{D})) = \{F \in \mathbb{H} : L(F) \in \varphi H^2(\mathbb{D})\},$$

where L is a determinantal operator.

The set Q_φ is a closed S-invariant subspace of $\mathbb{H}$.

Determinantal bricks

Let φ be either identically zero or an inner function.

A *determinantal brick* Q_φ *based on* φ is any set of the form

$$Q_\varphi := L^{-1}(\varphi H^2(\mathbb{D})) = \{F \in \mathbb{H} : L(F) \in \varphi H^2(\mathbb{D})\},$$

where L is a determinantal operator.

The set Q_φ is a closed S-invariant subspace of $\mathbb{H}$.

For $d = 1$, i.e., $\mathbb{H} = H^2(\mathbb{D})$, $Q_\varphi = \varphi H^2(\mathbb{D})$.

Determinantal spaces

A *determinantal space* $\mathcal{Q}_\varphi$ *based on* φ is the intersection of a finite number of determinantal bricks based on φ, that is,

$$\mathcal{Q}_\varphi = Q^1_\varphi \cap \ldots \cap Q^n_\varphi,$$

where Q^j_φ is a determinantal brick based on φ, $j = 1, \ldots, n$.

Determinantal spaces

A *determinantal space* $\mathcal{Q}_\varphi$ *based on* φ is the intersection of a finite number of determinantal bricks based on φ, that is,

$$\mathcal{Q}_\varphi = Q_\varphi^1 \cap \ldots \cap Q_\varphi^n,$$

where Q_φ^j is a determinantal brick based on φ, $j = 1, \ldots, n$.

Every determinantal space is a closed S-invariant space.

For $d = 1$, i.e., $\mathbb{H} = H^2(\mathbb{D})$, every determinantal space is a determinantal brick $\varphi H^2(\mathbb{D})$.

Invariant subspace for $S : \mathbb{H} \to \mathbb{H}$

Theorem (Br.-Gallardo Gutierrez, 2024)

Let $N \neq \{0\}$ be a closed subspace of $\mathbb{H}$. Then N is S-invariant if and only if there exist inner functions φ, ϕ and determinantal subspaces $\mathcal{Q}_\varphi$ and $\mathcal{Q}_0$ such that either

$$N = \phi\left(\mathcal{Q}_\varphi \cap \mathcal{Q}_0\right),$$

or

$$N = \phi \mathcal{Q}_\varphi.$$

Invariant subspace for $S : \mathbb{H} \to \mathbb{H}$

Theorem (Br.-Gallardo Gutierrez, 2024)

Let $N \neq \{0\}$ be a closed subspace of $\mathbb{H}$. Then N is S-invariant if and only if there exist inner functions φ, ϕ and determinantal subspaces $\mathcal{Q}_\varphi$ and $\mathcal{Q}_0$ such that either

$$N = \phi\left(\mathcal{Q}_\varphi \cap \mathcal{Q}_0\right),$$

or

$$N = \phi\mathcal{Q}_\varphi.$$

For $d = 1$, i.e., $\mathbb{H} = H^2(\mathbb{D})$, we recover Beurling's theorem.

Maximal subspace for $S : \mathbb{H} \to \mathbb{H}$

Corollary (Br.-Gallardo Gutierrez, 2024)

Let $N \subsetneq \mathbb{H}$ be a closed S-invariant subspace. Suppose that $\dim N^{\perp} \geq 2$. Then there exists a closed S-invariant subspace $M \subsetneq \mathbb{H}$ such that $N \subsetneq M$.

Maximal subspace for $S : \mathbb{H} \to \mathbb{H}$

Corollary (Br.-Gallardo Gutierrez, 2024)

Let $N \subsetneq \mathbb{H}$ be a closed S-invariant subspace. Suppose that $\dim N^\perp \geq 2$. Then there exists a closed S-invariant subspace $M \subsetneq \mathbb{H}$ such that $N \subsetneq M$. In particular, the invariant subspace problem has affirmative answer for operators of finite defect.

The Beurling-Lax matrix of a S-closed invariant subspace

Let M be a closed S-invariant subspace. By Beurling-Lax there is a S-inner operator $R : \mathbb{H} \to \mathbb{H}$ such that $M = R(\mathbb{H})$

The Beurling-Lax matrix of a S-closed invariant subspace

Let M be a closed S-invariant subspace. By Beurling-Lax there is a S-inner operator $R : \mathbb{H} \to \mathbb{H}$ such that $M = R(\mathbb{H})$

Lemma

There exists a $d \times d$ matrix $A = (a_{jk})$ whose entries are bounded holomorphic functions (actually of ∞-norm ≤ 1) such that

$$M = \overline{span\{p_1 \underline{a_1} + \ldots + p_d \underline{a_d}\}},$$

where $\underline{a_m} = a_{m1}e_1 + \ldots + a_{md}e_d$ and p_m are polynomials, $m = 1, \ldots, d$.

Idea of the proof in a particular case

Recall the (Beurling) inner-outer factorization of $f \in H^2(\mathbb{D})$: $f = IO$ where I is inner and O is outer.

Idea of the proof in a particular case

Recall the (Beurling) inner-outer factorization of $f \in H^2(\mathbb{D})$: $f = IO$ where I is inner and O is outer.

O *outer* if $O \in H^2(\mathbb{D})$ and $\overline{\text{span}\{S_1^n(O)\}} = H^2(\mathbb{D})$.

Idea of the proof in a particular case

Recall the (Beurling) inner-outer factorization of $f \in H^2(\mathbb{D})$: $f = IO$ where I is inner and O is outer.

O *outer* if $O \in H^2(\mathbb{D})$ and $\overline{\text{span}\{S_1^n(O)\}} = H^2(\mathbb{D})$.

I consider here only the case $d = 2$, $\det A \neq 0$ and **the (inner function) greatest common divisor of the inner parts of all not identically zero a_{jk} is** 1.

Idea of the proof in a particular case

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Idea of the proof in a particular case

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Define the determinantal operators $L_1, L_2 : \mathbb{H} \to H^2(\mathbb{D})$ by

$$L_1(f_1 e_1 + f_2 e_2) = \det \begin{pmatrix} f_1 & f_2 \\ b_1 & b_2 \end{pmatrix}$$

and

$$L_2(f_1 e_1 + f_2 e_2) = \det \begin{pmatrix} a_1 & a_2 \\ f_1 & f_2 \end{pmatrix}.$$

Idea of the proof in a particular case

Let φ be the inner part of $\det A$. Then

$$M = L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D})).$$

Idea of the proof in a particular case

Let φ be the inner part of $\det A$. Then

$$M = L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D})).$$

Let $\underline{a} = a_1 e_1 + a_2 e_2$, $\underline{b} = b_1 e_1 + b_2 e_2$.

Idea of the proof in a particular case

Let φ be the inner part of $\det A$. Then

$$M = L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D})).$$

Let $\underline{a} = a_1 e_1 + a_2 e_2$, $\underline{b} = b_1 e_1 + b_2 e_2$.

$$L_1(\underline{a}) = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \det A \in \varphi H^2(\mathbb{D})$$

Idea of the proof in a particular case

Let φ be the inner part of $\det A$. Then

$$M = L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D})).$$

Let $\underline{a} = a_1 e_1 + a_2 e_2$, $\underline{b} = b_1 e_1 + b_2 e_2$.

$$L_1(\underline{a}) = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \det A \in \varphi H^2(\mathbb{D})$$

$$L_2(\underline{a}) = \det \begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix} = 0 \in \varphi H^2(\mathbb{D}).$$

Idea of the proof in a particular case

Let φ be the inner part of $\det A$. Then

$$M = L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D})).$$

Let $\underline{a} = a_1 e_1 + a_2 e_2$, $\underline{b} = b_1 e_1 + b_2 e_2$.

$$L_1(\underline{a}) = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \det A \in \varphi H^2(\mathbb{D})$$

$$L_2(\underline{a}) = \det \begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix} = 0 \in \varphi H^2(\mathbb{D}).$$

Hence, $M \subseteq L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$

Idea of the proof in a particular case

Conversely, let $F = f_1 e_1 + f_2 e_2 \in L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$.

Idea of the proof in a particular case

Conversely, let $F = f_1 e_1 + f_2 e_2 \in L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$.
To prove that there exist polynomials $\{p_n\}$, $\{q_n\}$ such that

$$\|q_n \underline{a} + p_n \underline{b} - F\|_{H^2(\mathbb{D})} \to 0.$$

Idea of the proof in a particular case

Conversely, let $F = f_1 e_1 + f_2 e_2 \in L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$. To prove that there exist polynomials $\{p_n\}$, $\{q_n\}$ such that

$$\|q_n \underline{a} + p_n \underline{b} - F\|_{H^2(\mathbb{D})} \to 0.$$

We have $f_1 a_2 - f_2 a_1, f_1 b_2 - f_2 b_1 \in \varphi H^2(\mathbb{D})$.

Idea of the proof in a particular case

Conversely, let $F = f_1 e_1 + f_2 e_2 \in L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$.
To prove that there exist polynomials $\{p_n\}$, $\{q_n\}$ such that

$$\|q_n \underline{a} + p_n \underline{b} - F\|_{H^2(\mathbb{D})} \to 0.$$

We have $f_1 a_2 - f_2 a_1, f_1 b_2 - f_2 b_1 \in \varphi H^2(\mathbb{D})$.
By Beurling, there exist $\{p_n\}$, $\{q_n\}$ polynomials such that

$$\|q_n \det A - (f_1 a_2 - f_2 a_1)\|_{H^2(\mathbb{D})} \to 0, \quad \|p_n \det A - (f_1 b_2 - f_2 b_1)\|_{H^2(\mathbb{D})} \to 0.$$

Idea of the proof in a particular case

Hence, $\{b_1 q_n \det A\}$ converges in $H^2(\mathbb{D})$ to $b_1(a_2 f_1 - a_1 f_2)$ (since b_1 is bounded).

Idea of the proof in a particular case

Hence, $\{b_1 q_n \det A\}$ converges in $H^2(\mathbb{D})$ to $b_1(a_2 f_1 - a_1 f_2)$ (since b_1 is bounded).
Similarly, $\{a_1 p_n \det A\}$ converges in $H^2(\mathbb{D})$ to $a_1(b_2 f_1 - b_1 f_2)$.

Idea of the proof in a particular case

Hence, $\{b_1 q_n \det A\}$ converges in $H^2(\mathbb{D})$ to $b_1(a_2 f_1 - a_1 f_2)$ (since b_1 is bounded).
Similarly, $\{a_1 p_n \det A\}$ converges in $H^2(\mathbb{D})$ to $a_1(b_2 f_1 - b_1 f_2)$.
Thus, $\{\det A(-p_n a_1 + q_n b_1)\}$ converges to $f_1 \det A$.

Idea of the proof in a particular case

Hence, $\{b_1 q_n \det A\}$ converges in $H^2(\mathbb{D})$ to $b_1(a_2 f_1 - a_1 f_2)$ (since b_1 is bounded).
Similarly, $\{a_1 p_n \det A\}$ converges in $H^2(\mathbb{D})$ to $a_1(b_2 f_1 - b_1 f_2)$.
Thus, $\{\det A(-p_n a_1 + q_n b_1)\}$ converges to $f_1 \det A$.
Similarly, $\{\det A(-p_n a_2 + q_n b_2)\}$ converges to $f_2 \det A$.

Idea of the proof in a particular case

Hence, $\{b_1 q_n \det A\}$ converges in $H^2(\mathbb{D})$ to $b_1(a_2 f_1 - a_1 f_2)$ (since b_1 is bounded).
Similarly, $\{a_1 p_n \det A\}$ converges in $H^2(\mathbb{D})$ to $a_1(b_2 f_1 - b_1 f_2)$.
Thus, $\{\det A(-p_n a_1 + q_n b_1)\}$ converges to $f_1 \det A$.
Similarly, $\{\det A(-p_n a_2 + q_n b_2)\}$ converges to $f_2 \det A$.
Thus OK, if we can "divide" by $\det A$.

Idea of the proof in a particular case

$\det A = \varphi O$, with φ inner and O bounded and outer.

Idea of the proof in a particular case

$\det A = \varphi O$, with φ inner and O bounded and outer.
Multiplication by an inner function is an isometry in $H^2(\mathbb{D})$, thus we have

$$\|O[(-p_n a_1 + q_n b_1) - f_1]\|_{H^2(\mathbb{D})} \to 0,$$

and

$$\|O[(-p_n a_2 + q_n b_2) - f_2]\|_{H^2(\mathbb{D})} \to 0.$$

Idea of the proof in a particular case

Let $Z := L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$

Idea of the proof in a particular case

Let $Z := L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$
Thus $\overline{OZ} \subseteq \overline{OM}$.

Idea of the proof in a particular case

Let $Z := L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$
Thus $\overline{OZ} \subseteq \overline{OM}$.

Lemma

Let $N \subset \mathbb{H}$ be a closed S-invariant subspace. Let O be a bounded outer function. Then $\overline{ON} = N$.

Idea of the proof in a particular case

Let $Z := L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$
Thus $\overline{OZ} \subseteq \overline{OM}$.

Lemma

Let $N \subset \mathbb{H}$ be a closed S-invariant subspace. Let O be a bounded outer function. Then $\overline{ON} = N$.

Indeed, if O is outer, by Beurling we can find polynomials $\{r_n\}$ such that $\|r_n O - 1\|_{H^2(\mathbb{D})} \to 0$.

Idea of the proof in a particular case

Let $Z := L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D}))$
Thus $\overline{OZ} \subseteq \overline{OM}$.

Lemma

Let $N \subset \mathbb{H}$ be a closed S-invariant subspace. Let O be a bounded outer function. Then $\overline{ON} = N$.

Indeed, if O is outer, by Beurling we can find polynomials $\{r_n\}$ such that $\|r_n O - 1\|_{H^2(\mathbb{D})} \to 0$.
Hence $L_1^{-1}(\varphi H^2(\mathbb{D})) \cap L_2^{-1}(\varphi H^2(\mathbb{D})) = M$.

Determinantal spaces are not maximal

We sketch the case $Z = L_2^{-1}(\varphi H^2(\mathbb{D}))$, assuming $\dim Z^\perp > 1$.

Determinantal spaces are not maximal

We sketch the case $Z = L_2^{-1}(\varphi H^2(\mathbb{D}))$, assuming $\dim Z^\perp > 1$. We also assume here that $a_1 \neq 0$ let $I(a_1)$ be the inner part of a_1.

Determinantal spaces are not maximal

We sketch the case $Z = L_2^{-1}(\varphi H^2(\mathbb{D}))$, assuming $\dim Z^\perp > 1$.
We also assume here that $a_1 \neq 0$ let $I(a_1)$ be the inner part of a_1.
Let θ be the inner function such that $\varphi = \theta I(a_1)$.

Determinantal spaces are not maximal

We sketch the case $Z = L_2^{-1}(\varphi H^2(\mathbb{D}))$, assuming $\dim Z^\perp > 1$.
We also assume here that $a_1 \not\equiv 0$ let $I(a_1)$ be the inner part of a_1.
Let θ be the inner function such that $\varphi = \theta I(a_1)$.
Assume here that θ is not an automorphism nor constant.

Determinantal spaces are not maximal

We sketch the case $Z = L_2^{-1}(\varphi H^2(\mathbb{D}))$, assuming $\dim Z^\perp > 1$.
We also assume here that $a_1 \not\equiv 0$ let $I(a_1)$ be the inner part of a_1.
Let θ be the inner function such that $\varphi = \theta I(a_1)$.
Assume here that θ is not an automorphism nor constant.
Thus there exists a not constant inner function φ_1 which divides θ (and θ does not divide φ_1).

Determinantal spaces are not maximal

We sketch the case $Z = L_2^{-1}(\varphi H^2(\mathbb{D}))$, assuming $\dim Z^\perp > 1$.
We also assume here that $a_1 \neq 0$ let $I(a_1)$ be the inner part of a_1.
Let θ be the inner function such that $\varphi = \theta I(a_1)$.
Assume here that θ is not an automorphism nor constant.
Thus there exists a not constant inner function φ_1 which divides θ (and θ does not divide φ_1).
Let $N := L_2^{-1}(\varphi_1 H^2(\mathbb{D}))$.

Determinantal spaces are not maximal

We sketch the case $Z = L_2^{-1}(\varphi H^2(\mathbb{D}))$, assuming $\dim Z^\perp > 1$.
We also assume here that $a_1 \neq 0$ let $I(a_1)$ be the inner part of a_1.
Let θ be the inner function such that $\varphi = \theta I(a_1)$.
Assume here that θ is not an automorphism nor constant.
Thus there exists a not constant inner function φ_1 which divides θ (and θ does not divide φ_1).
Let $N := L_2^{-1}(\varphi_1 H^2(\mathbb{D}))$.

Determinantal spaces are not maximal

$$L_2(f_1 e_1 + f_2 e_2) = a_1 f_2 - a_2 f_1.$$

Determinantal spaces are not maximal

$$L_2(f_1 e_1 + f_2 e_2) = a_1 f_2 - a_2 f_1 .$$

Hence $Z \subset N$.

Determinantal spaces are not maximal

$$L_2(f_1 e_1 + f_2 e_2) = a_1 f_2 - a_2 f_1.$$

Hence $Z \subset N$.
On the other hand, $L_2(f e_2) = a_1 f$.

Determinantal spaces are not maximal

$$L_2(f_1 e_1 + f_2 e_2) = a_1 f_2 - a_2 f_1.$$

Hence $Z \subset N$.
On the other hand, $L_2(f e_2) = a_1 f$.
Thus, $e_2 \notin N$ (because $I(a_1)$ and φ_1 are coprime).

Determinantal spaces are not maximal

$$L_2(f_1 e_1 + f_2 e_2) = a_1 f_2 - a_2 f_1 .$$

Hence $Z \subset N$.
On the other hand, $L_2(f e_2) = a_1 f$.
Thus, $e_2 \notin N$ (because $I(a_1)$ and φ_1 are coprime).
Also, $\varphi_1 e_2 \in N \setminus Z$.

Determinantal spaces are not maximal

$$L_2(f_1 e_1 + f_2 e_2) = a_1 f_2 - a_2 f_1.$$

Hence $Z \subset N$.
On the other hand, $L_2(f e_2) = a_1 f$.
Thus, $e_2 \notin N$ (because $I(a_1)$ and φ_1 are coprime).
Also, $\varphi_1 e_2 \in N \setminus Z$.
Therefore, $Z \subsetneq N \subsetneq H$ and Z is not maximal.

Thank you