Picard-type extension theorems for unbounded target domains

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(joint work with **Annapurna Banik**)

Geometric Methods of Complex Analysis

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Part (1) of the above result with $Z = \mathbb{CP}^1, Y = \mathbb{C} \setminus \{0, 1\}$ is implied by the Big Picard Theorem. Hence, extension results of the above kind are called *Picard-type extension theorems.*

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- Picard-type extension problems — with bounded domains as target — become trivial because of Riemann's removable singularities theorem!

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Though $\mathscr{L}_\Omega$ needs a choice of def'n. function for its determination, two def'n. functions differ by a $\mathcal{C}^2$-smooth factor non-vanishing in a nbd. of $\partial\Omega$. Given the equation for $H_\xi(\partial\Omega)$ w.r.t. each def'n. function, Levi-form ineq. makes sense.

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Largeness of $k_\Omega(z; \mathbf{u})$ encodes the difficulty of finding "big" Ω-valued analytic discs tangent to the direction $\mathbf{u}$.

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- Our method relies on the regularity theory for the complex Monge–Ampère equation to derive lower bounds for k_Ω. Theorem 2 is the prototypical illustration of our method.

Technical input I: the visibility property

The raw idea of visibility involves a complete metric space (X, d):

(i) "nice" enough to admit geodesic lines through $x \neq y \in X$,

(ii) an abstract "boundary" $\mathsf{bd}(X)$ determined by d,

(iii) a topology on $\overline{X} := X \sqcup \mathsf{bd}(X)$ s.t. $\overline{X}$ is Hausdorff & the inclusion $\mathbf{j} : X \hookrightarrow \overline{X}$ is a homeo., **and**

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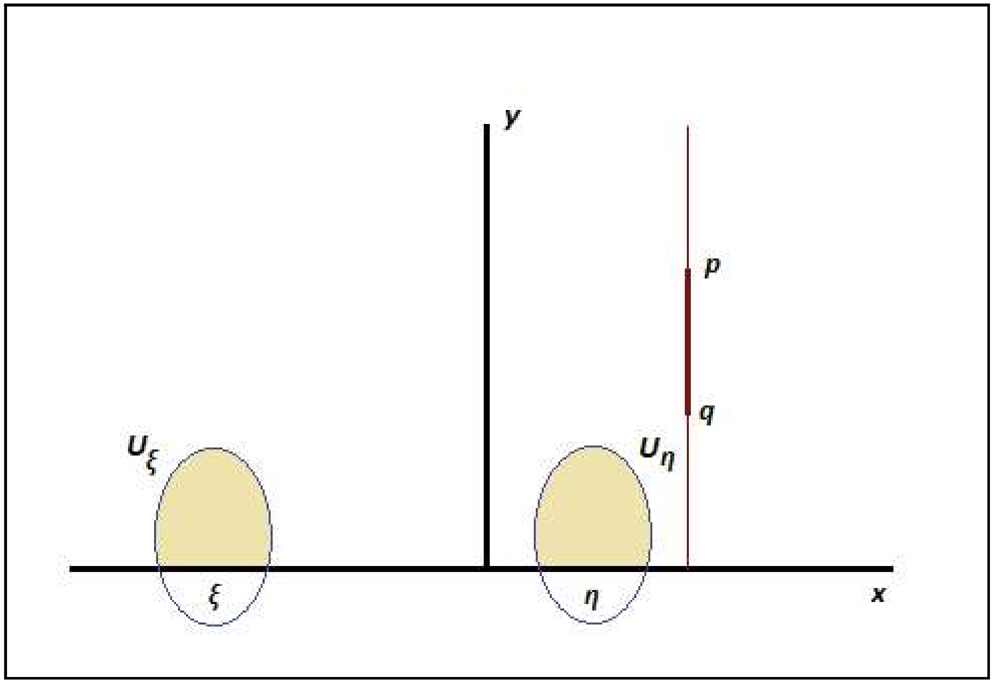

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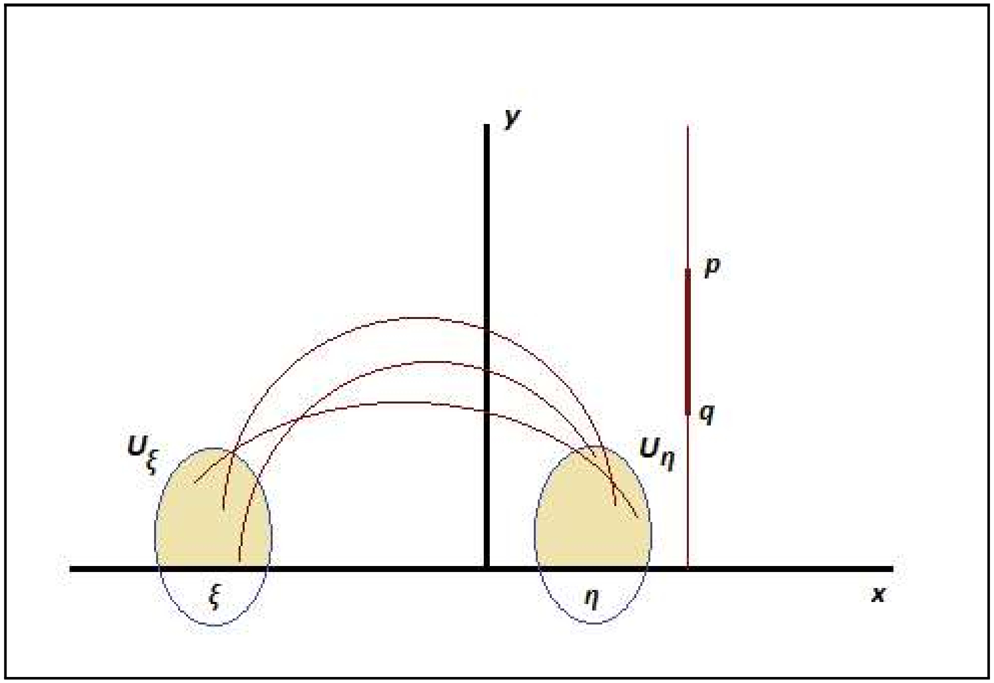

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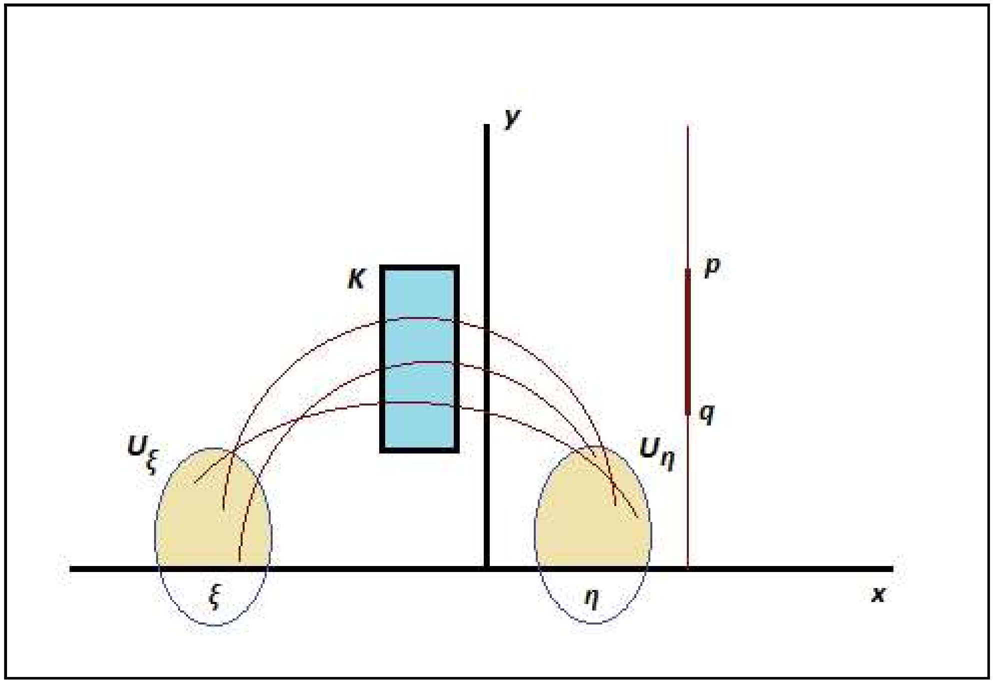

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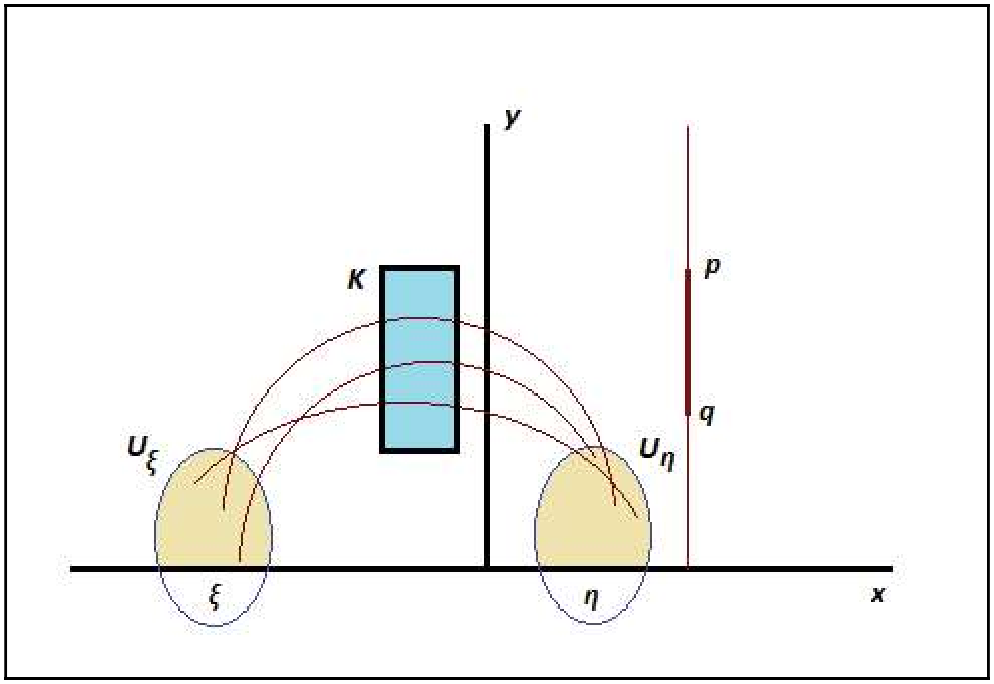

$(*)$ is seen, e.g. when $(X, \text{bd}(X))$ = (a CAT(0) space, its visual boundary).

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We adapt the above framework: it recognises the fact that for $(X, d) = (\Omega, K_\Omega)$, $\Omega \subsetneq_{\mathrm{dom}} \mathbb{C}^n$, **very** hard to estimate K_Ω and know if (Ω, K_Ω) is Cauchy-complete.

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A function $\omega : [0, \infty) \to [0, \infty)$ is called a *modulus of continuity* if it is concave, monotone increasing, and s.t. $\lim_{x \to 0^+} \omega(x) = \omega(0) = 0$.

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- For z close enough to $\partial\Omega$, $\exists \nu_z \geq \nu_0$ s.t. $1/2^{(\nu_z+1)} < \delta_\Omega(z) \leq 1/2^{\nu_z}$. For this z, easy to estimate ($\xi_z :=$ closest boundary point to z)
$$\begin{aligned} |U(z)| &\leq \omega(1/2^{\nu_z}) + |(\Phi(z) + \|z\|^2) - (\Phi(\xi_z) + \|\xi_z\|^2)| \\ &\leq \omega(1/2^{\nu_z}) + C_\phi\, \omega(\delta_\Omega(z)) + C_1 \delta_\Omega(z). \end{aligned}$$

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Theorem 2 hinges on a result by Ha–Khanh that says that *if* $\Omega \subset \mathbb{C}^n$, $n \geq 2$, *is a bdd. pscvx. domain having* $\mathcal{C}^2$*-smooth bdry., if* ρ *is a defining function of* Ω, *and* $m \geq 2$ *s.t.*

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- Ha–Khanh gives the modulus of cont. of ϕ to be $\approx t^{2/m}$. Now, just re-do the last proof with $Ct^{2/m}$ in place of $C\omega(t)$ for the right power of $\delta_\Omega(z)$. □

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- Let V_p be a nbd. of p that is so small that $\delta_{D_p}(z) = \delta_\Omega(z)\ \forall z \in V_p \cap \Omega$ and s.t. $(\partial V_p \cap \partial D_p) \Subset \partial D_p \cap \partial\Omega$.

The proof of Theorem 1, cont'd.

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The proof of Theorem 1, cont'd.

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- It follows from a result by Zimmer–B. that if, for each $p \in \partial\Omega$, the above estimate holds and, given $o \in \Omega$, $\exists C_p > 0, \alpha_p \geq 1/2$ s.t.

$$K_\Omega(o,z) \leq C_p + \alpha_p \log\big(1/\delta_\Omega(z)\big) \;\; \forall z \in V_p \cap \Omega,$$

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- The latter estimate is *almost* trivial because $\partial\Omega$ is $\mathcal{C}^2$-smooth. This establishes Step I.

The proof of Theorem 1, cont'd.

Step 2: Completing the proof

The proof of Theorem 1, cont'd.

STEP 2: Completing the proof

We need the following result:

PROPOSITION: Let $\Omega \subset \mathbb{C}^n$ be a Kobayashi hyperbolic domain and suppose $\partial\Omega$ is visible.

The proof of Theorem 1, cont'd.

STEP 2: Completing the proof

We need the following result:

PROPOSITION: Let $\Omega \subset \mathbb{C}^n$ be a Kobayashi hyperbolic domain and suppose $\partial\Omega$ is visible. Then, Ω is hyperbolically imbedded in $\mathbb{C}^n$.

We thus conclude from STEP I that Ω is hyperbolically imbedded in $\mathbb{C}^n$.

14 The proof of Theorem 1, cont'd.

Step 2: Completing the proof

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Proposition: Let $\Omega \subset \mathbb{C}^n$ be a Kobayashi hyperbolic domain and suppose $\partial\Omega$ is visible. Then, Ω is hyperbolically imbedded in $\mathbb{C}^n$.

We thus conclude from Step I that Ω is hyperbolically imbedded in $\mathbb{C}^n$. Then, mildly adapting a result by Joseph–Kwack — which removes the relative compactness constraint of Kiernan's result — our result follows. □

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14 The proof of Theorem 1, cont'd.

STEP 2: Completing the proof

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THANK YOU!