Segre and Chern forms associated with singular metrics

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Joint work in progress with Richard Lärkäng

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Segre and Chern forms of a Hermitian line bundle

If $E \to X$ Hermitian line bundle and e^{ψ} the metric in a local frame, then

$$c(E) = 1 - dd^c\psi, \quad s(E) = \frac{1}{c(E)} = 1 + dd^c\psi + (dd^c\psi)^2 + \cdots$$

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If $E' \to X$ the same line bundle but with metric φ', then $v = \varphi' - \varphi$ is global function and

$$dd^c v = c_1(E') - c_1(E) = c(E) - c(E').$$

Thus $c_1(E)$ and $s_k(E)$ determine Bott-Chern classes $\hat{c}_1(E)$ and $\hat{s}_k(E)$.

Vector bundle with singular metrics

In LRSW (based on LRRS) vector bundles $E \to X$ with Griffiths negative metric ($\log|\cdot|$ psh on the manifold E) with analytic singularities. They defined Segre and Chern forms $s(E)$ and $c(E)$.

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Definition

We say that the metric is *admissible* if $\log|\cdot|$ qpsh (quasi-psh) with analytic singularities.

Notice that if E is admissible, then any subbundle is admissible. Pullback of admissible bundle is admissible. Direct sums preserve admissibility, etc.

Example

Model case: Let E, F be vector bundles over X, $g\colon E \to F$ a holomorphic morphism. If F has a smooth Hermitian metric, then

$$|\alpha|^2 = |g\alpha|^2$$

is an admissible singular metric on E. □

This is not immediate but an argument is required.

Example

E is a line bundle, s_j global sections of E^*. The singular metric $|\cdot|e^{\psi}$ with $\psi = \log \sum_j |s_j|^2$ is of this kind. Take g as $\xi \mapsto \oplus_j \xi s_j$ and F a trivial bundle. □

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Besides LRRS and LRSW, we use ideas from AESWY and A

Main result

Let $E \to X$ be a vector bundle with an admissible singular metric, and let $E_0 \to X$ be the same bundle but with a smooth reference metric.

There is an associated Segre form $s(E)$ (in general a current) such that the following holds:

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(ii) $s(E)$ is the expected form where the metric is smooth (or sufficiently regular)

(iii) If $\pi \colon X' \to X$ is a modification, then $\pi^* E$ is admissible and

$$\pi_* s(\pi^* E) = s(E).$$

Main result, continued

(iv) $\mathrm{mult}_x s_k(E)$ are non-negative integers for each x and k (Lelong numbers if $s_k(E) \geq 0$). Independent of E_0.

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(v) For $k = 0, 1, 2, \dots$ we have (Siu type decomposition)

$$s_k(E) = \sum a_j [Z_j^E] + N_k^E$$

where Z_j have codim k

and $\mathrm{mult}_x N_k^E$ vanishes outside a set of codimension $\geq k+1$.

Chern and Segre forms, smooth metric, differential geometric definition

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$$c(E) = \det\left(I + \frac{1}{2\pi i}\Theta\right) = c_0(E) + c_1(E) + \cdots . \tag{0.1}$$

Obs $c_0(E) = 1$ so that

$$s(E) := 1/c(E) \tag{0.2}$$

is well-defined.

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A challenge to use (0.1) when metric singular so that Θ not smooth.

The idea in LRRS is to use definition in algebraic geometry:

Segre and Chern forms, smooth metric

$E \to X$ Hermitian vector bundle, in a local trivialization $E = X \times \mathbb{C}^r$. Let

$$p\colon \mathbb{P}(E) \to X$$

be the projectivization (there are two conventions) so that locally $\mathbb{P}(E) = X \times \mathbb{P}(\mathbb{C}^r)$.

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Consider $p^*E \to \mathbb{P}(E)$ with the tautological sub(line)bundle $L_E \to \mathbb{P}(E)$ so that the fiber over $(x, [\alpha])$ is the line $\{\lambda\alpha\}$ in $(p^*E)_{(x,[\alpha])} = \mathbb{C}^r$.

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L_E has an induced (smooth) metric and hence $s(L_E)$ is a smooth form on $\mathbb{P}(E)$.

Definition

$$s(E) := p_*(s(L_E)) \tag{0.3}$$

Obs a smooth form since p submersion.

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If

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(well-defined up to pluriharmonic), then

$$s(L_E) = \frac{1}{c(L_E)} = \frac{1}{1 - dd^c\psi} = \sum_{\ell=0}^{\infty} (dd^c\psi)^\ell.$$

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so that

$$s(E) = p_*\Big(\sum_{\ell=0}^{\infty} (dd^c\psi)^\ell\Big).$$

More suitable if ψ is non-smooth!

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so the Chern form is

$$c(E) := \frac{1}{s(E)} = \frac{1}{1 + (s(E) - 1)} = \sum_{k=0}^{\infty} (-1)^k (s(E) - 1)^k$$

(finite sum!)

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Lemma

If E is admissible, then ψ is qpsh with analytic singularities.

Analytic singularities means that locally $\psi = \log|s|^2 + bdd$, where s is a holomorphic tuple.

Definition of $s(E)$

$Z' \subset \mathbb{P}(E)$ the set where ψ is not locally bounded. Generalized Monge-Ampère products, defined recursively

$$[dd^c\psi]^0 = \mathbf{1}, \quad [dd^c\psi]^{\ell+1} = dd^c\big(\psi \mathbf{1}_{\mathbb{P}(E)\setminus Z'}[dd^c\psi]^\ell\big).$$

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Obs

$$\langle dd^c\psi\rangle^\ell = \mathbf{1}_{\mathbb{P}(E)\setminus Z'}[dd^c\psi]^\ell$$

is the non-pluripolar Monge-Ampère product.

Definition of $s(E)$ (continued)

We use the notation

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Let ψ_0 be associated with E_0. We define

$$s(E) = p_*\Big(\frac{1}{1-\langle dd^c\psi\rangle} + \frac{1}{1-dd^c\psi_0}\mathbf{1}_{Z'}\sum_{\ell=1}^{\infty}[dd^c\psi]^{\ell}\Big) \tag{0.4}$$

Segre forms as homomorphisms

The current $s(E)$ is a *quasi-cycle*, an element in the $\mathbb{Z}$-module $\mathcal{QZ}(X)$, an extension of the $\mathbb{Z}$-module of analytic cycles $\mathcal{Z}(X)$.

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$\mathcal{QZ}(X)$ is $\mathbb{Z}$-module generated by all currents of the form $\tau_*\gamma$, where $\tau\colon W \to X$ proper and

$$\gamma = dd^c b_1 \wedge \ldots \wedge dd^c b_t,$$

b_j qpsh bounded metrics on line bundles $L_j \to W$.

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Lemma

There is a homomorphism

$$\mathcal{QZ}(X) \to \mathcal{QZ}(X), \quad \mu \mapsto s(E)\mu,$$

such that $s(E)1 = s(E)$, $s_0(E) = I$, and $s(E)\mu$ is multiplication by $s(E)$ where it is smooth.

Chern form associated with an admissible singular metric

Since $s(E) - I$ has positive bidegree

$$c(E) = \sum_{\ell=0} (-1)^{\ell} (s(E) - I)^{\ell}$$

is a well-defined homomorphism $\mathcal{QZ}(X) \to \mathcal{QZ}(X)$ and

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Natural to define $c(E)$, acting on 1, as the Chern form of E.

Theorem

If E is admissible, the Chern form $c(E)$ has the same properties as $s(E)$ in the previous theorem.

Proof of Main result part (i)

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In $\mathbb{P}(E) \setminus Z'$ we have

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The residue is

$$\mathbf{1}_{Z'} dd^c\Big(v \frac{1}{1 - dd^c\psi_0} \frac{1}{1 - \langle dd^c\psi\rangle}\Big) = \mathbf{1}_{Z'} dd^c\Big(\psi \frac{1}{1 - dd^c\psi_0} \frac{1}{1 - \langle dd^c\psi\rangle}\Big) =$$

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Taking p_* we get (i).

An example

Example

Let $X = \mathbb{C}^2_x$, $E = X \times \mathbb{C}^2_\alpha$, $F = X \times \mathbb{C}^2$, with trivial metric, and E with the singular metric from $g\colon E \to F$ defined by

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \tag{0.5}$$

Then

$$s(E) = 1 + s_1(E) + s_2(E) = 1 + [x_1 x_2 = 0] + [x_1 = x_2 = 0].$$

□

Thank you for your attention!